Proceedings of the 8th European Conference of the Prognostics and Health Management Society 2024 - ISBN – 978-1-936263-40-0

Analytical Modeling of Health Indices for Prognostics and Health Management

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Abstract

Understanding the current health condition of complex systems and their temporal evolution is an important step in prognostics and health management (PHM). However, when managing a fleet of complex systems, variations arising from manufacturing, environmental factors, mission profiles, and maintenance practices result in diverse health index (HI) trajectories. Therefore, in PHM, it is essential not only to identify common fleet-wide trends but also to account for individual asset-level variations when inferring HIs.

While several data-driven approaches exist for inferring individual asset-level HIs from unsupervised run-to-failure degradation data (see e.g. (Djeziri, Benmoussa, & Zio, 2020)), little research has been devoted to deriving analytical probabilistic representations of HIs that encompass both fleet-level trends and individual asset-level fluctuations. This paper aims to bridge this gap by addressing the research question of how to obtain an analytical representation of probability distributions for the time to reach intermediate degradation levels, using run-to-failure data or incomplete degradation trajectories from a fleet of complex systems.

In this work, it is assumed that suitable, asset-specific HI curves have been inferred through a fusion of deep learning techniques and prior expert knowledge of degradation physics (e.g., (Bajarunas, Baptista, Goebel, & Chao, 2023)). Given this context, we derive an analytical probabilistic description of the health index (HI) that reflects both fleet-wide trends and asset-specific conditions in the cases of Gamma or Weibull time-to-failure (TTF) distributions. Our approach involves defining HIs with a power law function, enabling the modeling of TTF and time to reach intermediate degradation levels. Moreover, we also detail the procedure for estimating the power law exponent from field data through regression analysis and conduct a sensitivity analysis regarding this exponent.

To illustrate our methodology, we present two case studies based on the N-CMAPPS dataset of turbofan engines and Li-ion batteries, validating the aforementioned assumptions and illustrating our methodology steps.

1. Introduction

An important step in prognostics and health management of complex industrial systems is inferring their current health condition. To this end, a normalized health index is often defined as a metric that measures the degree of degradation of equipment. Conventionally, a value of 1 for the health index corresponds to perfect health, and a value of 0 to a failed state. An intermediate value characterizes a state where the item is still operating but less than perfectly. If the health index captures the physical condition of the asset correctly, the time evolution of the health index is an appropriate means for performing prognostics, i.e., predicting the evolution of a degradation, eventually up to a failure, and the time until that failure, or remaining useful life (RUL). Therefore, the health index for an asset constitutes a key tool for maintenance decision-making, as it enables health assessment (in particular, degradation severity) and prognostics.

The derivation of HIs has traditionally depended on extracting key features from condition monitoring (CM) data and integrating them with a physical understanding of the asset to create a health index (Atamuradov et al., 2020). This practice, while effective, is heavily reliant on domain-specific knowledge, presenting a significant barrier to scalability and adaptability across different systems. To address these limitations, diverse data-driven approaches have been proposed for estimating HI from condition monitoring data. For instance, supervised learning models have been applied when dealing with datasets that contain labels of HIs (Roman, Saxena, Robu, Pecht, & Flynn, 2021). Similarly, residual techniques that identify deviations from a system's expected behavior (Ye & Yu, 2021; Hsu, Frusque, & Fink, 2023) offer another pathway, albeit contingent on the existence of a representative dataset of "healthy" state labels- an often challenging prerequisite in industrial settings due to difficulties in obtaining a representative data for complex systems. Recently, unsupervised methods combining deep learning methodologies with traditional reliability engineering principles in the form of explicit, analytical representation of the health index have shown promise in inferring asset-specific HI (Bajarunas et al., 2023; Yang, Habibullah, & Shen, 2021; Qin, Yang, Zhou, Pu, & Mao, 2023). Therefore, these recent works highlight the potential of leveraging the extensive body of reliability engineering theory, alongside deep learning algorithms, to model RUL dynamics effectively. An in-depth study of RUL dynamics and uncertainty, based on reliability theory, is reported in (Dersin, 2023).

In this work, our objective is to provide a theoretical foundation for constructing a robust analytical HI that reflects both fleet-wide trends and asset-specific conditions. By doing this, we aim to enable the integration of reliability engineering models in machine learning algorithms by providing an analytical probabilistic description of the HI. Addressing this objective involves answering the following question: *How to find an analytical description for a time-dependent health index integrating random parameters to capture asset variability and align with observed times to reach various degradation severity levels including the time to failure?*

Hence, in this work, we assume the availability of time-to-failure (TTF) distributions for a fleet of assets. Given this assumption, we formulate the problem in a general context and provide an analytical solution when the TTF follows a Gamma distribution or a Weibull distribution. In this scenario, with a health index defined by a power law featuring either an inverse-Gamma or a Fréchet-distributed coefficient, as the case may be, we demonstrate that the time to reach any intermediate degradation level follows a Gamma or Weibull distribution, respectively, sharing the same shape parameter as the TTF. Moreover, the scale parameter explicitly depends on the degradation level. We also detail the procedure for estimating the power law exponent from field data through regression analysis and conduct a sensitivity analysis regarding this exponent.

To validate our methodology, we present case studies focusing on the N-CMAPPS turbofan and randomized usage Li-ion batteries datasets. The results confirm the proposed methodology and highlights its practical applications. Obtaining an explicit, analytical representation of the health index, including the random variability among assets, is a definite advance over the state of the art that offers a major advantage. The proposed approach enables maintenance decision-making with minimal computational demand.

The paper is organized as follows: Section 2 presents the methodology used in this work; we first formulate the problem in Section 2.1 and present a resolution method in Section 2.2. We then delve into specific cases involving Gamma (Section 2.3) and Weibull distributions (Section 2.4), followed by a discussion on estimating the power law exponent controlling the shape of degradation for both analyzed distributions (Section 2.5) . Case studies from the N-CMAPSS and randomized battery usage datasets illustrate our approach (Section 3), with sensitivity analysis on the power law exponent (Section 4). The paper concludes with a summary of our findings and suggestions for future research in Section 5.

2. METHOD

This section provides a detailed explanation of the methodology used to derive an analytical description of the HI. The process is divided into several steps, which are outlined below and visually summarized in Figure 1

2.1. Problem Statement

A degradation phenomenon can be described by an HI, which evolves with time i.e., $HI(t)$, usually monotonically, from a perfect health condition to a failed state. Perfect health corresponds to a value $HI(t) = 1$, and failure is deemed to occur at the first time t, for which $HI(t)$ hits 0.

Given a plausible probability distribution for the time to failure, denoted T, which is derived from available data or prior knowledge, it is desired to find a family of probability distributions for the times T_s needed for the HI to reach any intermediate health level s,

$$0 < s < 1 \tag{1}$$

In other words, given a prior probability distribution, conditional upon $HI(0) = 1$, for the time to failure T,

$$T = \inf[t : HI(t) = 0] \tag{2}$$

find, for any intermediate level s, the probability distribution for T_s:

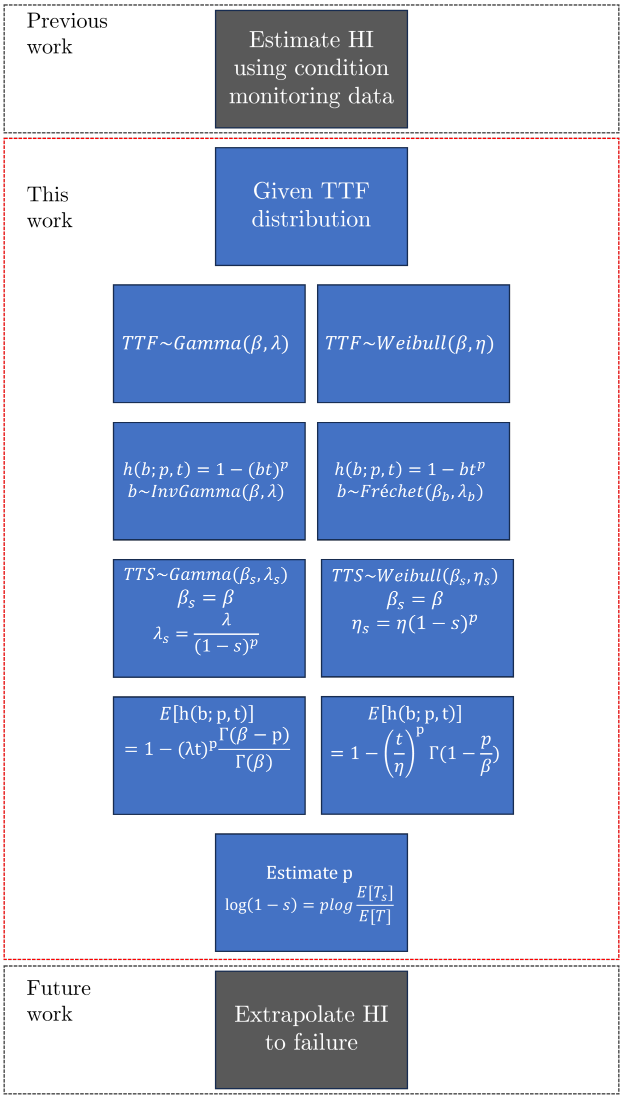


Figure 1. Flowchart illustrating the different steps of the proposed method along with the context of its applicability. (*Previous work*) The methodology assumes the availability of asset-specific Health Index (HI) derived from CM data, for instance, based on Bajarunas et al. (*This work*) The process of estimating begins with the assumption of a prior plausible Time to Failure (TTF) distribution (Gamma or Weibull models) and a probabilistic parametric model of a Health Index ($h(b,p,t)$). Based on these assumptions, we derive the analytical forms of the distributions describing the time required for the HI to reach any specified intermediate health level (TTS), thus providing a comprehensive statistical framework to model a degradation process. (*Future work*) The estimation of probabilistic, asset-specific failure times through extrapolation of individual HI's is suggested as a possible application of the derived analytical HI.

$$T_s = \inf[t : HI(t) = s] \tag{3}$$

2.2. Resolution Method: General Principle

Let $R(t)$ denote an assumed reliability function. Then a probabilistic model for $HI(t)$, as a non-increasing function of t, is selected, and the condition $P[T > t] = R(t)$ is imposed. Finally, Eq. (3) is applied to obtain the distribution of T_s:

$$R_s(t) = P[T_s > t] \tag{4}$$

Let us consider the following parametric model for the health index:

$$HI(t) = h(p_1, p_2, ...p_n; t) \tag{5}$$

with an assumed functional form h, where some of the parameters $p_1, p_2, \ldots, p_n$ are random variables.

Then, it should be noted that

$$h(p_1, p_2, ...p_n; t) > 0 \tag{6}$$

is equivalent to

$$T > t \tag{7}$$

therefore the following condition is imposed:

$$P[h(p_1, p_2, ...p_n; t) > 0] = R(t) \tag{8}$$

with the right-hand side of Eq. (8) known.

Similarly, the condition $T_s > t$ is equivalent to $HI(t) > s$ and hence from Eq. (8), one derives

$$P[h(p_1, p_2, ..., p_n; t) > s] = R_s(t) \tag{9}$$

for any value of s between 0 and 1.

The method is quite general and can be applied to any TTF distribution. In the next two subsections, the method is detailed and illustrated on two frequently encountered families of TTF distributions: Gamma and Weibull, respectively.

2.3. Gamma Case

Let us consider the case when the time to failure follows a Gamma distribution with shape parameter β and rate parameter λ. The Gamma reliability function for time T (Nachlas, 2017) can be expressed as:

$$R(t) = 1 - \frac{\gamma(\lambda t; \beta)}{\Gamma(\beta)} \tag{10}$$

where $\gamma(\lambda.t; \beta)$ stands for the incomplete Euler gamma function.

A health index is sought, $HI(t)$, such that the time for the HI to reach the value 0 is Gamma-distributed.

We shall now show that a solution is provided by the following power law for the health index:

$$h(b; p; t) = 1 - (bt)^p \tag{11}$$

with a positive exponent p and a random variable b with an inverse-gamma distribution with shape parameter β and scale parameter λ (b has the dimension of a frequency, i.e., the inverse of a time, so does λ). The health index defined by Equation (11) decreases monotonically from 1 to 0 as the time or usage variable t increases from 0 to $\frac{1}{b}$. It is a convex function of t if $p < 1$ and a concave function if $p > 1$ (and linear in the limit case of $p = 1$). The property that b has an inverse-Gamma distribution is equivalent to $\frac{1}{b}$ having a Gamma distribution with parameters β (shape) and λ (rate).

Denoting by T the time to failure, there follows from the above health index definition that

$$P[T > t] = P[(bt)^p < 1] = P[bt < 1] = P[\frac{1}{b} > t] \quad (12)$$

Since $\frac{1}{b}$ is Gamma distributed , the right-hand side of (12) is the Gamma reliability function at time t, with shape and rate parameters respectively equal to β and λ. Therefore, it has been proved that the definition (11) for the health index leads to a Gamma-distributed time to failure. .

Now let us look at the distribution of the time for the health index to reach a level s, between 0 and 1.

Let us denote that first hitting time T_s.

$$P[T_s > t] = P[h(b;p;t) > s] = P[1 - (bt)^p > s] \quad (13)$$

Equation (13) is equivalent to:

$$P[T_s > t] = P[(bt)^p < 1 - s] = P[\frac{1}{b} > \frac{t}{(1-s)^{\frac{1}{p}}}] \quad (14)$$

Since $\frac{1}{b}$ is Gamma (β,λ) distributed, it follows from (10) that,

$$R_{T_s}(t) = P[T_s > t] = 1 - \frac{\gamma(\frac{\lambda t}{(1-s)^{\frac{1}{p}}}; \beta)}{\Gamma(\beta)} \quad (15)$$

Therefore it has been shown that T_s has a Gamma distribution with shape factor β, and rate parameter λ_s given by the following function of s and the exponent p:

$$\lambda_s = \frac{\lambda}{(1-s)^{\frac{1}{p}}} \quad (16)$$

The problem stated in the beginning has thus been solved in the case when the time to failure has a Gamma distribution. The mathematical expectations of T_s and that of the health index $HI(t)$ are then derived explicitly, as follows, from the properties of the gamma distribution and the inverse-gamma distribution (Llera & Beckmann, 2016):

$$E(T_s) = \frac{\beta}{\lambda_s} = \frac{\beta}{\lambda}(1-s)^{\frac{1}{p}} \quad (17)$$

which can also be written as :

$$E(T_s) = E(T)(1-s)^{\frac{1}{p}} \quad (18)$$

To derive the expectation of the health index $HI(t)$; we now use properties of the inverse-gamma distribution. If X has an inverse-gamma distribution with parameters β and λ, the nth-order moment of X is given (Llera & Beckmann, 2016) by:

$$E(X^n) = \lambda^n \frac{\Gamma(\beta - n)}{\Gamma(\beta)} \quad (19)$$

as long as

$$n < \beta$$

Therefore

$$E[HI(t)] = 1 - E(b^p)t^p = 1 - (\lambda t)^p \frac{\Gamma(\beta - p)}{\Gamma(\beta)} \quad (20)$$

assuming the exponent p to be smaller than the shape factor β.

2.4. Weibull Case

We shall now consider the case where the time to failure follows a 2-parameter Weibull distribution. Denoting β and η the shape and scale parameters, respectively, this corresponds to the well-know reliability function:

$$R(t) = e^{-(t/\eta)^\beta} \quad (21)$$

For the health index, let us take the following power law, slightly different from the one taken in the Gamma distribution case, for reasons which will become apparent:

$$h(b;p;t) = 1 - bt^p \quad (22)$$

where p is a positive exponent, and b is a random variable. It will be seen that, if b has a Fréchet distribution (Fréchet, 1927; Ramos, Louzada, Ramos, & Dey, 2020), then the time to failure is Weibull distributed.

Indeed, by definition of the Fréchet (also known as "inverse Weibull") distribution, if the random variable b is Fréchet-distributed with scale parameter λ_b and shape parameter β_b, then:

$$P[b > u] = 1 - exp[-(\frac{u}{\lambda_b})^{-\beta_b})] \tag{23}$$

The shape parameter β_b is dimensionless, and the scale parameter λ_b has the dimension of t to the power of $(-p)$, just as the coefficient b.

Then, by substituting

$$u = t^{-p} \tag{24}$$

in (23), the following is obtained :

$$P[HI(t) > 0] = P[b < t^{-p}] = exp[-(\lambda_b t^p)^{\beta_b}] \tag{25}$$

and this expression must be equated to $P[T > t]$, which is assumed to be the reliability function of a two-parameter Weibull variable (η ,β).

Therefore, the parameters of the Fréchet distribution for b are obtained as follows:

$$\lambda_b = 1/\eta^p \tag{26}$$

$$\beta_b = \beta/p \tag{27}$$

as it can be verified by substituting the right-hand sides of (26) and (27) respectively for λ_b and β_b in (25). Then the distribution of T_s, the first hitting time of level s, can be derived as well, for any value of s between 0 and 1.

$$P[Ts > t] = P[HI(t) > s] \tag{28}$$

$$= P[1 - bt^p > s] = P[b < (1 - s)t^{-p}] \tag{29}$$

Therefore, by substituting for u in (23) the value $(1 - s)t^{-p}$ and using (26) and (27),

$$P[Ts > t] = exp[-(t^p/\eta^p(1 - s))^{\frac{\beta}{p}}] \tag{30}$$

or

$$P[Ts > t] = exp[-(1 - s)^{-\frac{\beta}{p}}(\frac{t}{\eta})^{\beta}] \tag{31}$$

It is seen that (31) describes the reliability function of a Weibull random variable with: 1) the same shape factor β as the distribution of T; 2) A scale factor η_s expressed as follows as a function of s , the scale factor η of T and the exponent p:

$$\eta_s = \eta(1 - s)^{\frac{1}{p}} \tag{32}$$

Thus, the problem stated in the beginning has also been solved in the Weibull distribution case.

Accordingly, the mathematical expectation of the first hitting time T_s is obtained:

$$E(T_s) = \eta(1 - s)^{\frac{1}{p}}\Gamma(1 + \frac{1}{\beta}) \tag{33}$$

Equation (33) can also be formulated as

$$E(T_s) = E(T)(1 - s)^{\frac{1}{p}} \tag{34}$$

which is the same as in the Gamma-distribution case (18). Also, the expectation of the health index $HI(t)$ at time t can be derived from the expectation of the random coefficient b, assumed Fréchet distributed:

$$E(b) = \frac{1}{\eta^p}\Gamma(1 - \frac{p}{\beta}) \tag{35}$$

Therefore

$$E(HI(t)) = 1 - E(b)t^p = 1 - (\frac{t}{\eta})^p\Gamma(1 - \frac{p}{\beta}) \tag{36}$$

The quantiles of b can also be derived. The x-percent quantile is B_x:

$$B_x = \frac{1}{\eta^p(-lnx)^{\frac{p}{\beta}}} \tag{37}$$

In particular, the median (50-percent quantile) is given by:

$$B_{0.5} = \frac{1}{\eta^p(ln2)^{\frac{p}{\beta}}} \tag{38}$$

2.5. Estimation of Exponent p from Data

From (32), there follows, by taking logarithms,

$$\log(1 - s) = p\log(\frac{\eta_s}{\eta}) \tag{39}$$

Therefore, after estimating η_s from the data sample for various values of s, the regression coefficient of $\log(1 - s)$ with respect to $\log(\frac{\eta_s}{\eta})$ will provide an estimation of p. Also, taking (34) into account,

$$log(1 - s) = p\log(\frac{E(T_s)}{E(T)}) \tag{40}$$

Therefore, in order to estimate p, it is equivalent to estimate $E(T_s)$ from the data samples corresponding to several values of s and then run the linear regression of $\log(1 - s)$ with respect to $\log(\frac{E(T_s)}{E(T)})$. The regression coefficient (slope) is the best estimate of p. The same method applies in the Gamma distribution case since the dependence of E(T_s) on s is the same in both cases (see Section 2.3).

2.6. Incomplete Degradation Trajectories

Our method for obtaining an analytical form of the HI does not require run-to-failure condition monitoring data [1]. Let us

[1] If no failures are observed the HI has a different meaning as it is normalized with respect to the most degraded unit in the fleet.

consider u as the smallest threshold of $HI(t)$ observed for all units in the fleet. Then in equation (34), instead of considering the expected TTF, $E(T)$, we would consider the expected time to hit the common threshold $E(T_u)$. The revised equation would be:

$$E(T_s) = E(T_u)\frac{(1-s)^{1/p}}{(1-u)^{1/p}} \tag{41}$$

Where $E(T_s)$ is the sample arithmetic mean for each value $s > u$. When $u = 0$, this is equivalent to Eq. 34. The exponent p can be estimated from linear regression in

$$\log(1-u) - \log(1-s) = p(\log(E(T_u)) - \log(E(T_s))) \tag{42}$$

3. Case Studies

3.1. Turbofan

The New Commercial Modular Aero-Propulsion System Simulation (N-CMAPSS) dataset (Arias Chao, Kulkarni, Goebel, & Fink, 2021) offers comprehensive degradation trajectories of turbofan engines until failure. Among the dataset's eight subsets, we focus on DS003, characterized by a failure mode impacting the efficiency and flows of both low-pressure and high-pressure turbines.

The N-CMAPSS dataset characterizes degradation at the component level across initial, normal, and abnormal degradation stages. Consequently, an HI is calculated through a non-linear mapping of operational margins under reference conditions. System failure is determined when the HI reaches 0. The dataset also accounts for between-flight maintenance by allowing improvements in engine health parameters within specified limits. The ground truth HI is shown in Figure 2, and will be used to verify the findings of Section 2.3 and 2.4. Estimating the HI using condition monitoring data as highlighted in (Bajarunas et al., 2023) is also possible.

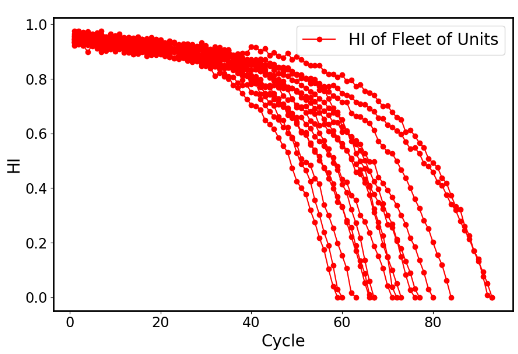


Figure 2. Observed HI in N-CMAPSS DS03 Dataset

The Akaike Information Criterion (AIC) (Akaike, 1974) was used to compare the goodness of fit with different probability distributions (Weibull, Gamma, Exponential), see Table 1. When a statistical model is used to represent the process that generated some data, some information is lost. The AIC, based on information theory, estimates the amount of information lost. It deals both with overfitting and underfitting by taking model simplicity into account as well as goodness of fit. The AIC is defined by

$$AIC = -2log(maxL) + 2P \tag{43}$$

where the term log(maxL) denotes the maximum value of the log-likelihood function, and P is the number of parameters in the model (for instance,for Weibull or Gamma, P is equal to 2). In our example, the best value of the AIC was obtained with the Gamma distribution for the time to failure as well as the time to reach level s for s ranging from 0 to 0.8. The AIC value for Weibull distribution is almost identical. In contrast, the AIC value for the exponential distribution is much higher.

Using the Maximum Likelihood Estimation technique, we estimated the best-fit Gamma parameters for various s thresholds. Figure 3 shows the estimated β_s and λ_s values for $s = [0, 0.1, 0.2, ..., 0.8]$. The results validate the conclusion presented in Section 2.3: the distribution of the first hitting time T_s shares the same shape factor $\beta = 52.83$ as the distribution of failure times T. Additionally, the rate parameter λ_s is a function of s and λ of T. We determined $p = 3.35$ following the description provided in Section 2.5. The wide confidence intervals of β_s and λ_s can be primarily attributed to the limited number of observations (15 run-to-failure curves), rather than to the choice of the Gamma distribution, which we have demonstrated to be the most suitable among the alternative distributions investigated.

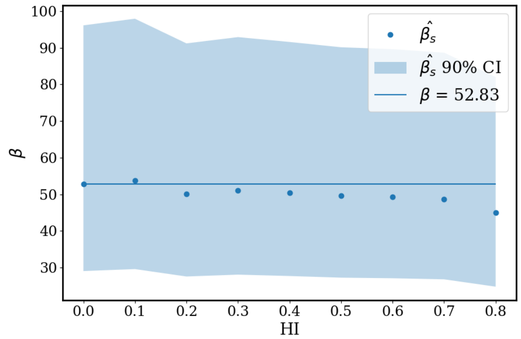
We then estimated the best-fit Weibull parameters for various s thresholds. In Figure 4, we estimated β_s and η_s using $s = [0, 0.1, 0.2, ..., 0.8]$. The results validate the conclusion presented in Section 2.4: the distribution of the first hitting time T_s shares the same shape factor $\beta = 7.32$ as the distribution of failure times T. Additionally, the scale parameter η_s is a function of s and η of T.

Figure 5 illustrates the mean, median, and 90% quantile of $HI(t)$, as described by equations (36) and (37). Notably, we observe that the median closely aligns with the ground truth HI within the dataset.

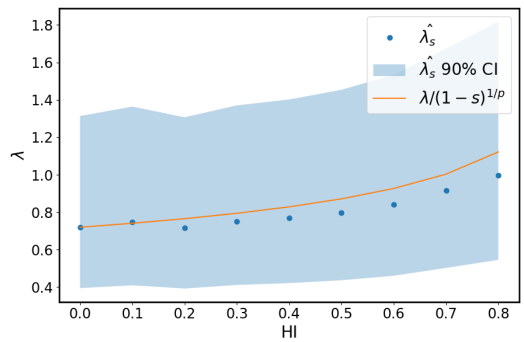
3.2. Battery

The methodology proposed in this study was further validated using a dataset obtained from the NASA Ames Prognostics Center of Excellence repository, specifically focusing on battery usage patterns (Bole, Kulkarni, & Daigle, 2014). This dataset includes information collected from individual 18650 LCO cells undergoing various charging and discharging cycles following randomized protocols.

Batteries commonly exhibit several physical aging mecha-



(a) β_s



(b) λ_s

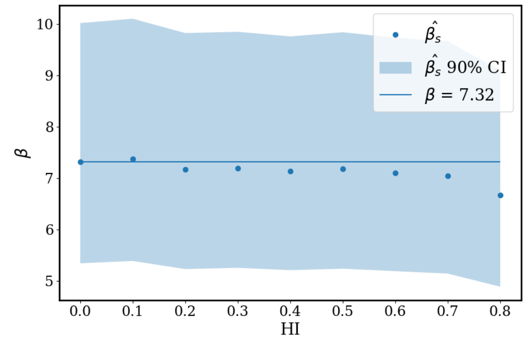
Figure 3. The Gamma distribution shape factor β_s and the rate parameter λ_s for various HI thresholds for N-CMAPSS dataset.

Table 1. AIC of distribution fits for CMAPSS turbofan case study.

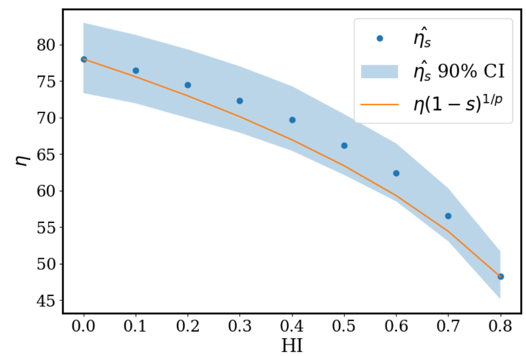
s	AIC Gamma	AIC Weibull	AIC Exponential
0	118	120	161
0.1	117	120	161
0.2	117	120	160
0.3	116	119	159
0.4	115	118	158
0.5	114	116	156
0.6	112	114	154
0.7	109	112	151
0.8	106	108	147
0.9	116	113	127

nisms such as graphite exfoliation, electrolyte loss, solid electrolyte interface layer formation, continuous thickening, and lithium plating, among others (Sui et al., 2021). These aging processes lead to two primary changes in battery behavior: capacity degradation and increased internal resistance. In this analysis, our focus will be on capacity degradation as the key health index for the batteries under investigation.

The HI of a battery is defined as the ratio between its current capacity and the nominal capacity (Q/Q_{nominal}). The battery's capacity can be determined by reference discharge cycles conducted at a constant current (I). The current capacity is calculated as the integral of current over the entire



(a) β_s



(b) η_s

Figure 4. The Weibull distribution shape factor β_s and the scale factor η_s for various HI thresholds for N-CMAPSS dataset.

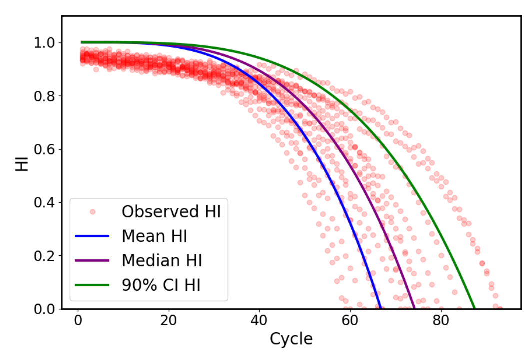


Figure 5. The mean, median, and 90% quantile of the health index obtained from Weibull distribution.

reference discharge cycle, denoted as $\int_t I$.

In this work, the failure of a battery (HI = 0) is defined once the capacity ratio is less than 60%. The initial HI of the battery is equal to the initial capacity ratio. Figure 6 shows the estimated HI of the NASA battery dataset.

The AIC values of three different distribution fits are shown in Table 2. The best fit was obtained with Gamma distribution for the time to failure as well as the time to reach level s for s ranging from 0 to 0.9. The AIC value for Weibull dis-

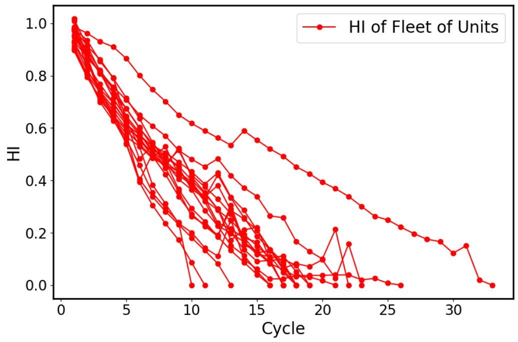
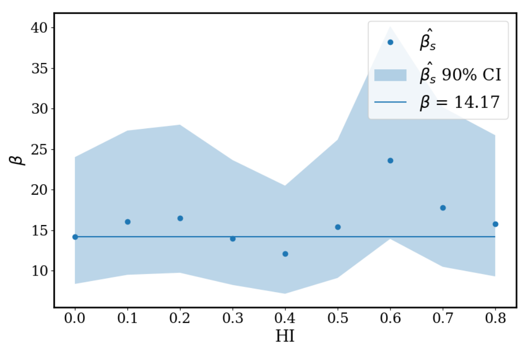


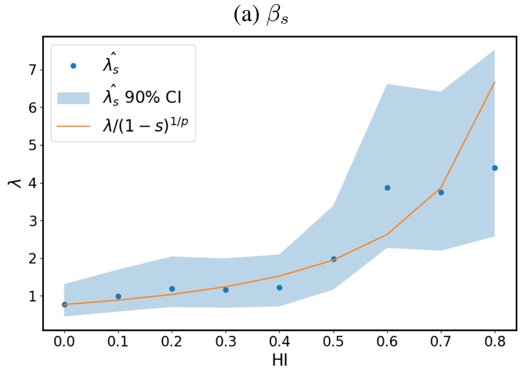
Figure 6. Observed HI in NASA battery dataset

tribution is almost identical,in contrast with the exponential distribution AIC, much higher.

We employed the Maximum Likelihood technique to estimate the best-fit Gamma parameters for various s thresholds. In Figure 7, we estimated β_s and λ_s using $s = [0, 0.1, 0.2, ..., 0.8]$. Once more, we illustrate that a reasonably good approximation for the shape parameter β_s of the first hitting time is the shape parameter β of T. Following the estimation of $p = 0.94$, we demonstrate that the rate parameter λ_s varies with s and λ. Since $p < 1$, the HI curve is now convex, as observed.



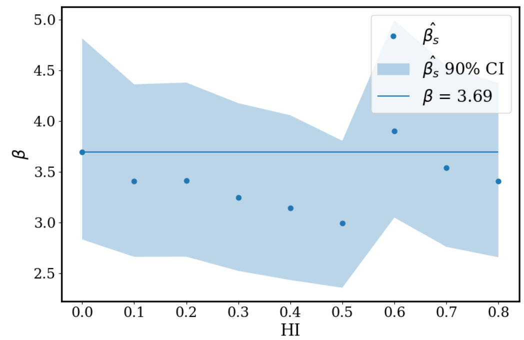
(a) β_s



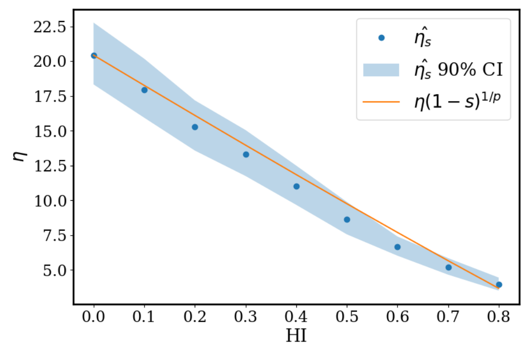
(b) λ_s

Figure 7. The Gamma distribution shape factor β_s and the rate parameter λ_s for various HI thresholds for NASA battery dataset.

The best-fit Weibull parameters for various s thresholds are shown in Figure 8. Once again, we show that a reasonably good approximation for the shape parameter of the first hitting time β_s is the shape parameter β of the failure time T and that the scale parameter η_s varies with s and η as expected.



(a) β_s



(b) η_s

Figure 8. The Weibull distribution shape factor β_s and the scale factor η_s for various HI thresholds for NASA battery dataset.

Table 2. AIC of other distribution fits for NASA battery case study.

s	AIC Gamma	AIC Weibull	AIC Exponential
0.0	118	121	151
0.1	111	118	146
0.2	104	112	140
0.3	102	108	135
0.4	97	102	128
0.5	84	93	118
0.6	67	75	109
0.7	62	70	99
0.8	54	60	89
0.9	44	50	71

4. Sensitivity Analysis

Sensitivity analysis has been conducted on the N-CMAPSS dataset, to investigate the effect of the exponent p in the parametric model of the health index.

For the Gamma case, it is immediate from (16) that, for given

s, λ_s is a decreasing function of p (for p greater than, or equal to 1). In the limit of p going to infinity, λ_s converges to λ. For the Weibull case, a similar conclusion is drawn, but instead from (32) it follows that, for given s, η_s is an increasing function of p.

From (18) and (34) it follows that for both considered distributions the average time to reach threshold s, $E(T_s)$, is an increasing function of p, as illustrated in Figure 9.

For both distributions, when p increases, the average value of the HI is first higher than, and subsequently (for greater values of the time variable t), lower than, the HI corresponding to a lower value of p. Increasing p corresponds to delaying the decrease in HI, i.e., delaying the onset of the degradation; but, once the degradation occurs, it is more sudden. See Figure 10 for an illustration.

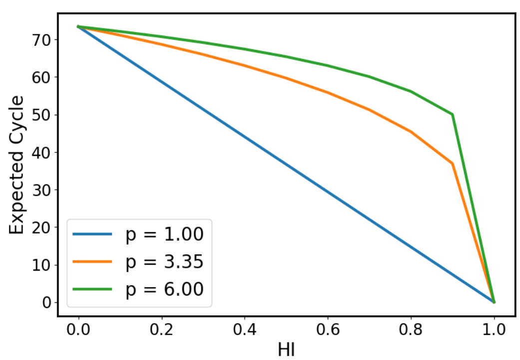
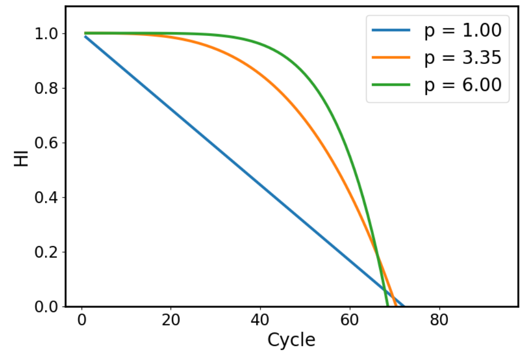
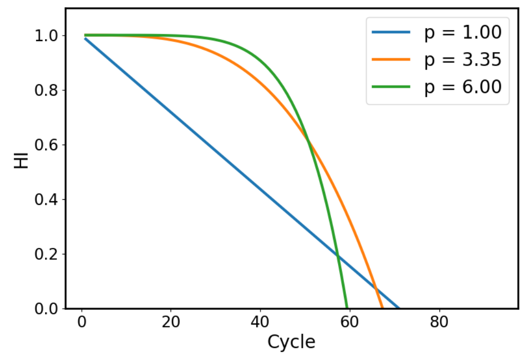


Figure 9. Gamma and Weibull distribution $E[T_s]$ as a function of s for three values of p. N-CMAPSS dataset.



(a) Gamma distribution $E[HI(t)]$ as a function of t for three values of p. N-CMAPSS dataset.



(b) Weibull distribution $E[HI(t)]$ as a function of t for three values of p.

Figure 10. Sensitivity to various p for the turbofan case study. N-CMAPSS dataset.

5. Conclusion and Perspectives

This study has successfully addressed the problem of analytically modeling health indices (HI) in cases where the time-to-failure follows either a Gamma or Weibull distribution. By leveraging observed health index trajectories over time and specifically the failure times, we have derived an analytical form for the health index that is consistent with these observations. Additionally, we provided an analytical expression for the distribution of the time to reach any intermediate degradation level.

The availability of closed-form expressions for the health index is highly beneficial for implementing predictive maintenance strategies, particularly for estimating the remaining useful life (RUL) distribution. Furthermore, once a health index function is derived for a particular application, it can potentially serve as a foundation for similar applications, such as the same asset under different operating conditions or a slightly modified asset. Without an analytical characterization, a new health index would need to be learned from scratch for each new dataset.

Future work could extend this approach to other TTF distributions and other HI formulations by applying the general methodology outlined in Section 2.2. Additionally, an important extension of this work could be the use of quantile regression and extrapolation of the HI from individual degradation trajectories. More broadly, the analytical health index approach represents a significant advancement in survival analysis, offering opportunities to integrate machine learning techniques, particularly 'deep survival' methods, with traditional reliability engineering.

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