# A multivariate statistical approach to the implementation of a health monitoring system of mechanical power drives


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## ABSTRACT

The implementation in service of accelerometric health monitoring systems of mechanical power drives has shown that a considerable number of false failure alarms is generated. The paper presents a combined application of several multivariate statistical techniques and shows how a monitoring method which integrates these tools can be successfully exploited in order to improve the reliability of the diagnostic systems.


## 1. INTRODUCTION

Failure diagnostics via condition monitoring on mechanical systems and components is a broad and very relevant topic. Different approaches based on the development of specific sensors and data-driven methods have been applied. For example in (K. Liu, 2013) is described the construction of a composite health index through the fusion of multiple sensor data. In many cases the calibration of reliable data-driven models is obstructed by the lack of data regarding the failure modes of the mechanical system. In such circumstances sophisticated anomaly detection and decision mechanisms might be required (see for example (Ramasso & Gouriveau, 2010)).

Our activity was performed under research contract granted by AgustaWestland. It was focused on monitoring the health conditions of mechanical power drives of helicopters. Accelerometric monitoring systems have been previously installed on several types of helicopters produced by AgustaWestland. The adopted vibration monitoring methods are based on analyzing analog signals provided by a set of accelerometers (we refer the reader to (Randall, 2011) and especially (CAA-PARER-2011/01, 2012)). Each power drive is monitored by a single accelerometer. The accelerometric outputs undergo Fourier spectral decomposition and the description of the local (not global) properties of the energy distribution through the spectrum of vibrational modes leads to a set of scalar health indicators, which are supposed to detect specific damages. For example relevant physical indicators represent the energy of the spectral components corresponding to the main rotational frequency and its multiples, the energy contained in a localised energy bands etc. Other indicators, obtained from the second-level signal analysis, are related to local variations, correlations between specific spectral channels, local shape factors and signal standard deviations. The monitoring methodology of the health state of a component is based on fixed critical thresholds for the values of each condition indicator and damage alerts are generated when *any* of the indicators exceeds the threshold for certain number of measures. In other words the adopted monitoring method concerns a univariate (independent) *interpretation* of the health indicators.

The implementation of this health monitoring system on power drives in actual service has shown that a considerably high number of false alarms is generated, thereby requiring additional troubleshooting workload.

The purpose of our research is to develop a health monitoring method able to reduce to the very minimum the false positives. The efficiency of the existing diagnostic systems has been improved via third-level multivariate treatment of the condition indicators. A monitoring method which integrates several multivariate statistical techniques has been developed and implemented. The method is able distinguish with very high level of statistical confidence true failure situations and false anomaly alerts if these have been previously observed and diagnosed on any other aircraft of the same type.

## 2. EXPERIMENTAL SETUP

Our research was focused on mechanical power drives of helicopters which consist of an assembly of several gears rotating on shafts supported by ball and roller bearings. AgustaWestland provided a large amount of data collected on sixteen aircrafts of the same type flying in different conditions. Our experimental data set consists of several thousands of measurements of the condition indicators of each mechanical component and was collected over a period of several months and hundreds of flight hours. Our study mainly concerned the following set of power drives in which true (confirmed by inspection of the power drive) and false alerts were detected: TTO Pinion, characterised by twelve condition indicators (CI), IGB Pin (12 CI's), TGB Gear (12 CI's), TRDS (2 CI's), 2nd Stage Pin RH Brgs (6 CI's), Oil cooler Brg (6 CI's), Hangar Ball Brg (9 CI's).

In some cases (TRDS and the Hangar Ball Brg) the single-valued thresholds of several health indicators were strongly exceeded in a false alert state and a true damage provoked a more moderate reaction of the monitoring system. These cases were considered as particularly "critical" as the monovariate evaluation of the damage appears to be misleading.

In the rest of the article the set of $N$ health indicators of a mechanical power drive will be interpreted as an element in a real $N$-dimensional vector space and called the *vector state* of the power drive.

## 3. MULTILINEAR RE-CALIBRATION AND ANOMALY DETECTION

The values of the standard health indicators, which characterise the normal operational regime of a mechanical component vary quite consistently between different aircrafts of the same type. If compared to each-other, the vector states of the same component in ordinary regime on different helicopters form well-distinguished clusters inside the vector space of indicators (a striking illustration is given on Fig. 1).

The fact that ordinary operational states of a power drive installed on different aircrafts cannot be compared, makes impossible the calibration of any sort of statistical model, based on historical collection of vector states measured on a fleet of helicopters. Moreover the mechanical components selected for our investigation are typically subject to a very low number of failures. A calibration and a validation of a reliable multivariate model on *each single aircraft* appears therefore as extremely unrealistic.

Besides the set of component vector states, a historical collection of simultaneous measurements of the following parameters of operational condition of each aircraft was available: Engine 1 Torque, Engine 2 Torque, Rotor Speed, Roll Angle, Pitch Angle, True Airspeed, Radio Altitude, Vertical Speed, Normal Acceleration, Density Altitude, Tail Rotor Torque, Main Rotor Torque, Roll Rate, Pitch Rate, Yaw Rate, Longitudinal Acceleration.

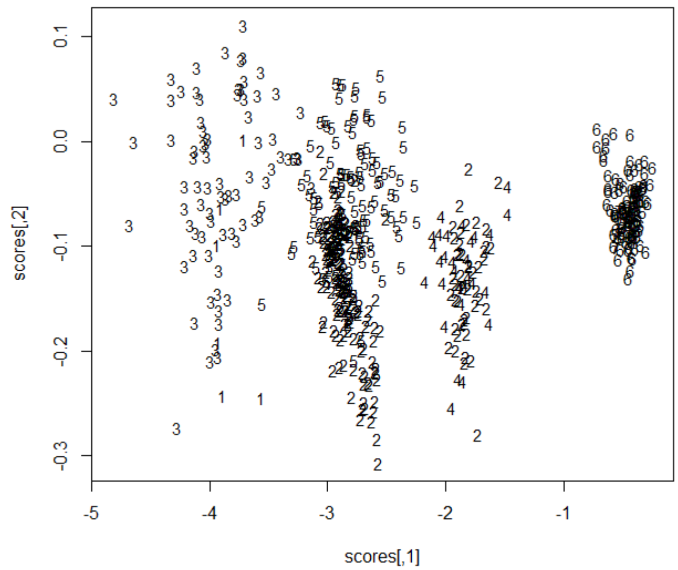


Figure 1. PCA scores of normal operational states of TRDS component on different helicopters (the marker is the "helicopter number").

It has been hypothesised that the accelerometric measurements are influenced by the environmental state of the aircraft. In order to test that hypothesis, *canonical correlation analysis* has been applied on the available data set. It has been observed that many components are characterised by three or four canonical correlations with considerably high values (over 0,5). This fact is quite relevant with respect to the interrelations between the environmental vector state and the component vector state. Unlikely in some cases (Hangar Ball Brg) the canonical correlation profile is characterised by high first (considered as accidental) and very low second canonical correlation.

The established multi-correlation between the aircraft states and component states led us to the construction of the following linear filter. A liner map $f : R^{17} \longrightarrow R^N$ (where $N$ is the dimension of the component vector) which provides a "predicted" component vector state in correspondence to each environmental state has been calibrated. The k-th row of the matrix associated to this linear map represents the coefficients of a multiple liner regression of the k-th component of the power drive vector over the set of environmental parameters.

If we compare Fig. 1 to Fig. 2, we observe that as a consequence of re-calibration, scores of normal operational states measured on different helicopters slightly concentrate and mix together quite uniformly. Furthermore the shape of the cluster of projections on the space generated by the first two principal components becomes more ellipsoidal. This means that

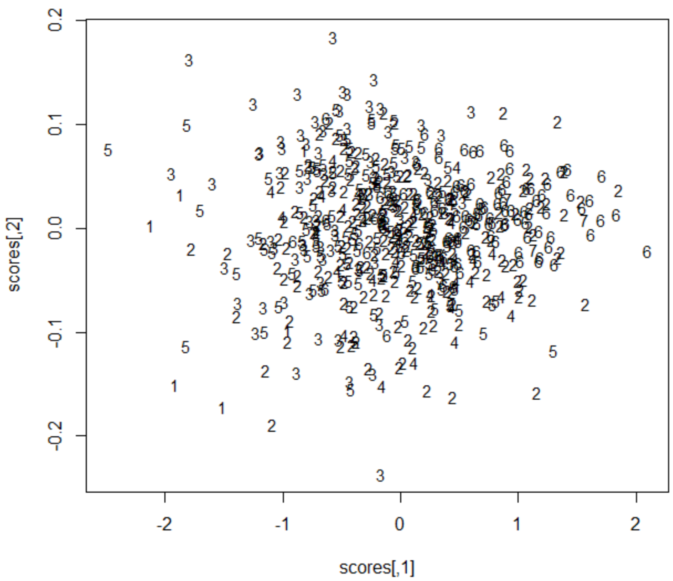


Figure 2. PCA scores of normal operational states of TRDS of different helicopters after linear re-calibration.

the linear re-calibration procedure filters the deterministic impact of the general state of the aircraft onto the accelerometric measurements. Once filtered the influence of the specific exploiting regime of the aircraft, the variability of the normal operational states of each mechanical component can be attributed to a random noise process. In other words, the filtered normal operational states of each power drive fit with a multidimensional Gauss distribution. This fact was verified by various multivariate normality tests like Kolmogorov-Smirnoff, Jarque-Bera etc. (see (Kolmogorov, 1936; A. Justel, 1997; C. M. Jarque, 1987)). It has been observed that both the distributions of filtered normal operational states of a component of a single helicopter and the filtered normal operational states of a component installed on different helicopters can be considered as Gaussian with a very high level of statistical confidence (p-value around $2 \times 10^{-15}$).

Similar effects are observed for all the mechanical components, for which the canonical correlation analysis reveals considerable level of linear correlation. Linear re-calibration makes vector states measured on different helicopters of the same type *comparable*. A specific situation on an aircraft can be compared to analogous situation on another aircraft.

The fact that filtered normal operational states of the power drives are normally distributed, enables us to implement a standard anomaly detection method based on the Mahalanobis distance, i.e. the multidimensional Shewhart control chart (see (Shewhart, 1931) and (Shewhart, 1986)).

A Shewhart control chart has been calibrated on the set of ordinary operational states of each mechanical component on a single helicopter. A small portion (less than 2%) of ordinary vector states exceed the control limit. The same control chart was applied to normal operational states of the same power drive, installed on other helicopters and bigger portion of states was judged out of control (15% for the Hangar Ball Brg). This means that even though linearly filtered data are used, there are still residual differences in the ordinary regime of mechanical components of different aircrafts. The same control chart has been also validated in the context of anomalous situations occurred on the same helicopter with very good results. In the case of Hangar Ball Brg roughly 73% of the states were judged as anomalous.

In conclusion, anomaly detection method based on a Shewhart control chart must be calibrated on each single helicopter. A software tool implementing a multivariate *self-learning* Shewhart control chart, which calibrates itself automatically on the ordinary regime of a single mechanical component and highlights anomalous states, has been produced. The program computes automatically the upper control limit by means of a Gaussian approximation of the Fisher-Snedecor distribution.

In many cases (especially TRDS and Hangar Ball Brg) the Mahalanobis distance between states corresponding to false alerts and the mean value of the normal regime exceeds the distance of the true damage states. For this reason the multivariate self-learning Shewhart control chart is an excellent tool for the detection of anomalous situations, but it is not sufficient for the discrimination of true failure states and anomaly alerts which do not correspond to a failure. Thus, additional discrimination statistical tools, as described later, have been applied.

## 4. METHOD

The linear re-calibration strongly reduces the differences between the normal operational regime of power drives installed on different aircrafts. This fact enables us to apply a set of standard multivariate statistical methods on a historical database of a fleet of helicopters. For a detailed description of those techniques we refer the reader to the following texts (Ferrell, 1979; Rencher, 2002; Timm, 2002; W. K. Härdle, 2012; Izenman, 2008).

We adopt a geometric viewpoint on multivariate statistics, since in our study an Euclidean approach provides some very useful intuitions on multivariate methods (see, on this aim (Wickens, 1995) and (Epps, 1993)). See also (Tyurin, 2009), where is presented a more intrinsic (coordinate free) geometric prospective on multivariate statistics. In this context we developed our analysis in terms of projections onto relevant subspaces. Our approach interprets (analogously but independently on (Gniazdowski, 2013)) correlations as angles and further radicalises this viewpoint by identifying statistical variables in terms of real projective classes in the space of random vectors.

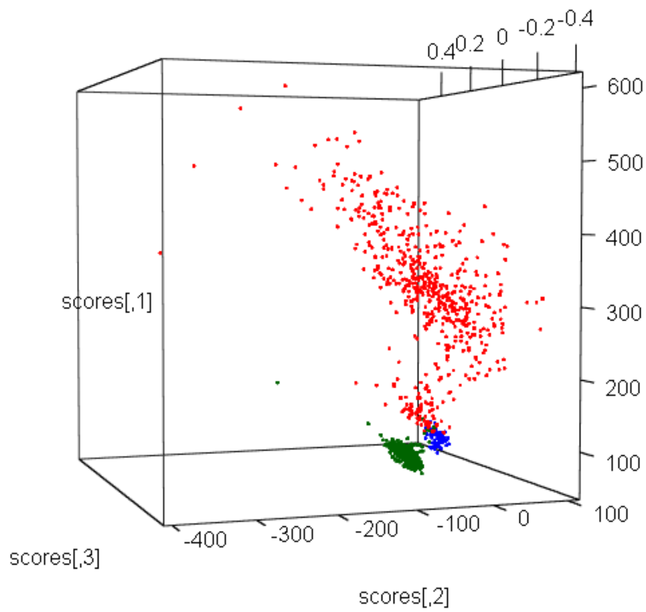


Figure 3. PCA scores of the states of a 2nd Stage Pin RH Brgs.

### 4.1. Structure of variance

The complete set of available states (normal, true failures, false alerts) of each mechanical component was processed by Principal Component Analysis (PCA). This technique highlights existing spontaneous clusterings in the variance structure of the data set. On Fig. 3 is displayed an example of scores of complete data sets on the subspace generated by the first three principal components. In this and in each of the following figures green dots represent scores of normal operational states, yellow orange and blue dots represent scores of false alert states and red dots - true failure vector states.

In the "critical case" of Hangar Ball Brg the projections on the subspace generated by the second and the third principal components reveal a relevant spontaneous clustering of the vector states.

PCA leads to a consistent dimensional reduction in the space of states. Equations of linear and quadratic separation surfaces between the projections of the group clusters have been easily worked out and simple control methods can be based on the spontaneous clustering.

The structure of variance in the data sets has been further explored by applying multivariate discrimination methods like Liner Discriminant Analysis (LDA) and Quadratic Discriminant Analysis (QDA)(see (W. K. Härdle, 2012)). The set of component state vectors has been divided into three groups, ordinary operational states, false alerts and true failures.

On Fig. 4 are displayed projections of TGB Gear states onto the subspace generated by the first three linear discriminant

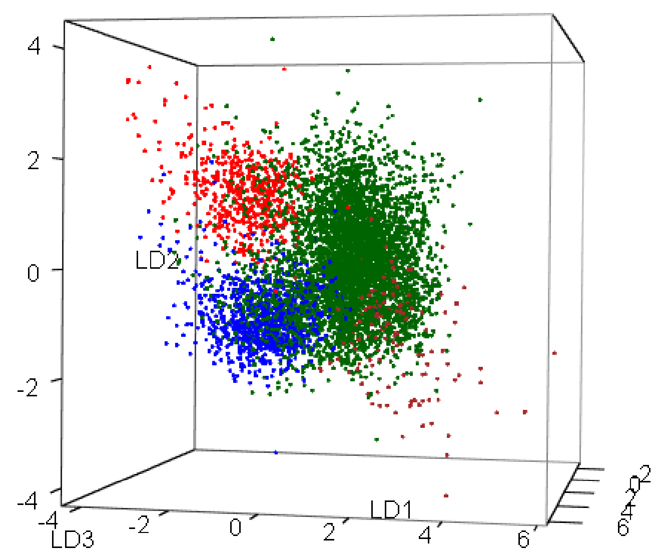


Figure 4. LDA scores of TGB Gear.

Table 1. Leave-one-out LDA re-classification of 2nd Stage Pin RH Brg vector states

| real \ classified as | false alert | normal | true failure |
|---|---|---|---|
| false alert | 74 | 0 | 0 |
| normal | 1 | 1869 | 0 |
| true failure | 8 | 67 | 495 |

functions.

The calibrated linear discriminant models were validated by standard leave-one-out procedure using the complete data set of the fleet. On Table 1, and Table 2 are displayed some examples of LDA re-classification results.

There is a well-known quadratic classifier based on the minimisation of the Mahalanobis distance (with some corrections) (see (Rencher, 2002)). On Table 3 and Table 4 ere displayed some examples leave-one-out quadratic discriminant validation results.

The results obtained by both LDA and QDA leave-one-out cross validation are quite encouraging, especially because of the small portion of miss-classified true failure states. In the "critical" case of the Hangar Ball Brg both methods provide statistically significative number of correctly classified true failure states. This means that true failure can be unambiguously detected.

### 4.2. Failure detection via canonical correlation

Canonical correlation analysis can be employed for detecting anomalies. Suppose that the ordinary operative regime of a process is characterised by a strong correlation between vector variables $X$ and $Y$. In such case one estimates the values of $Y$ starting from known values of $X$ by a suitable linear

Table 2. Leave-one-out LDA re-classification of Hangar Ball Brg vector states

| real \ classified as | false alert | normal | true failure |
|---|---|---|---|
| false alert | 54 | 6 | 4 |
| normal operat. | 29 | 1513 | 20 |
| true damage | 5 | 49 | 117 |

Table 3. Leave-one-out QDA re-classification of 2nd Stage Pin RH Brg vector states

| real \ classified as | false alert | normal | true failure |
|---|---|---|---|
| false alert | 74 | 0 | 0 |
| normal | 0 | 1860 | 10 |
| true failure | 0 | 0 | 570 |

model. If $Y$ assumes “unexpected” values i.e. its behaviour contrasts with the established correlation, this fact can be considered as a manifestation of some anomaly.

In our study, has been tested the hypothesis that anomalous behaviour of a mechanical component is uncorrelated with the environmental data. We would expect that the linear correlations between the environmental parameters and the components health indicators should decrease in presence of anomalous behaviour of the component. Therefore the data sets of normal states and data sets containing anomalous states have been compared in order to establish whether the relevant (high) linear correlation coefficients decrease.

The situation which emerges from this procedure appears slightly chaotic. For the TRDS the linear correlation is very strong and the values of the coefficients drastically drop in mixed regime which contains true failure states. For the IGB pin the linear correlation is strong, the correlations in mixed regime get certainly worse, but monitoring of that component did not give evidence for real failures, so the measured anomalies correspond to false alerts. The TGB gear is characterised by relatively high values of the significant correlation coefficients and its mixed regime contains a true failure, but it seems that the second canonical correlation slightly improves in mixed regime.

In conclusion, for components for which the linear correlation with the environmental states is particularly high our theoretical hypothesis is confirmed. This means that for those components the canonical correlation method can be considered as a supplementary anomaly detection resource.

### 4.3. Structure of covariance

In our study, a particular behaviour of the covariance matrix of the vector states of some mechanical components in case of anomalous measurements has been observed. The states of true damage are often characterised by increased correlation of certain vector components. The behaviour of the correlation matrix appeared slightly different in the case of false

Table 4. Leave-one-out QDA re-classification of Hangar Ball Brg vector states

| real \ classified as | false alert | normal | true failure |
|---|---|---|---|
| false alert | 60 | 2 | 2 |
| normal operat. | 63 | 1430 | 69 |
| true damage | 2 | 33 | 136 |

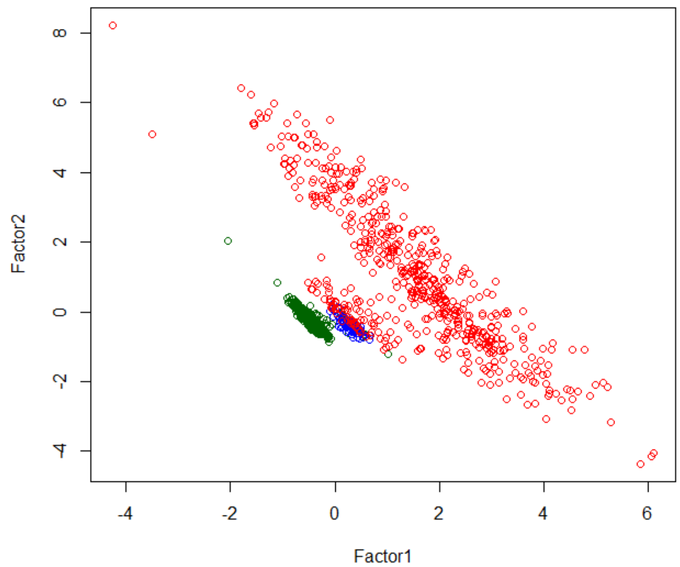


Figure 5. Bartlett factor scores of the 2nd Stage Pin RH Brgs.

anomaly reports.

A possible explanation of this phenomenon could be given if in the case of true failure, different health indicators react simultaneously in a consistent and correlated way (failure states provoke an enhancement of certain elements of the correlation matrix). On the contrary false alerts can be interpreted as anomalous measurements not necessarily induced by a consistent reaction of the monitoring system.

Canonical factor models have been calibrated on the set of state vectors. Typically the calibration of factor model based on two factors was possible, but in some cases (Hangar Ball Brg) the iterative procedure does not converge with two but with three factors.

In terms of projections onto the space generated by the principal factors, our hypothesis translates in the following way. We expect that the projections of the normal operational cluster (near by the origin) and true failure cluster (away from the origin) onto the subspace generated by the principal factors show different characteristic profiles. The direction in which failure states projections spread away from the origin is indicative regarding the correlation modifications introduced by the simultaneous reaction to a damage. The shape of the cluster of ordinary operational states characterises the intrinsic covariance structure of the component. In this context we

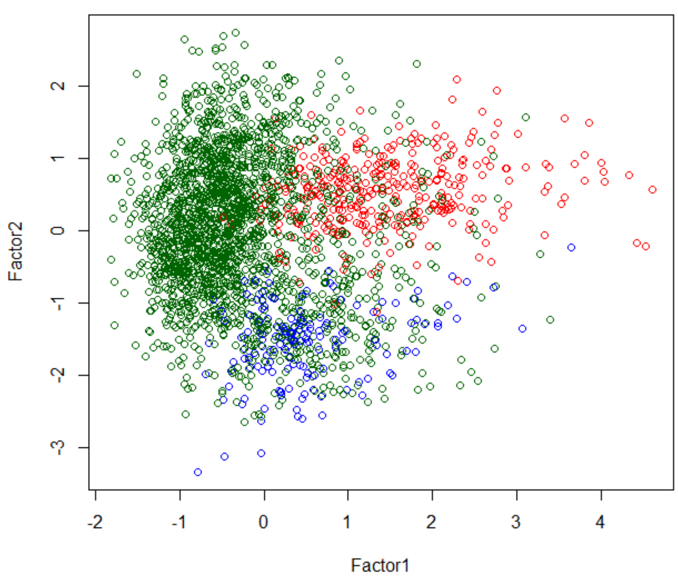


Figure 6. Bartlett factor scores of TGB Gear.

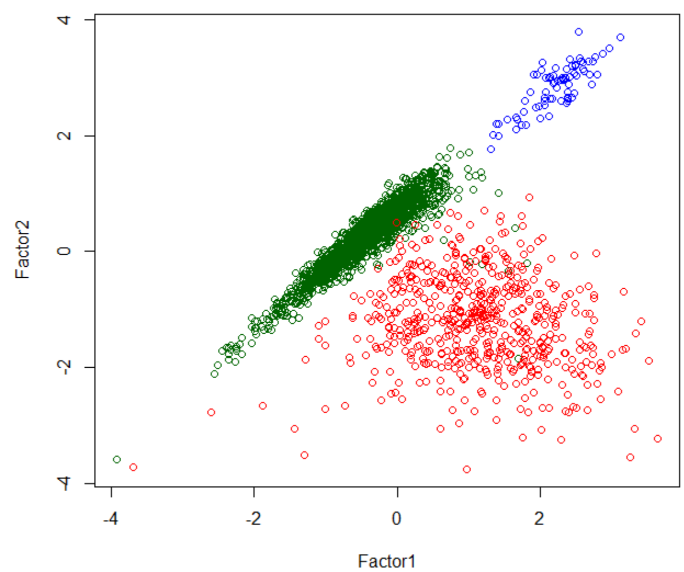


Figure 7. Bartlett type scores of unit states of a 2nd Stage Pin RH Brgs.

expect that anomalous or false alerts should reveal some sort of irregular behaviour.

On Fig. 5 and Fig. 6 are shown the projections of the states of the 2nd Stage Pin RH Brgs and the TGB gear. Clustering is present in both cases. Projections (factor scores) of true failure states spread away from the origin in a direction, which is characteristic for the modified covariance structure. Our study substantially confirmed our theoretical hypothesis. It is easy to work out linear or quadratic decision boundaries on factor scores.

In the case of Hangar Ball Brg the factor scores of the ordinary operational states concentrate again near by the origin and the anomalous states spread far from it. Nevertheless these projections do not reveal a striking separation between true and false alert states.

We conclude that for some mechanical components, the covariance structure of the vector data set provides further resources for defining discriminant procedures.

## 5. SPHERICAL STRUCTURE OF DATA SETS

Since (latent) variables was considered as real projective classes, we have hypothesised that the correlation structure of the data set can be better understood in terms of directions of the state vectors. In this context the module of a vector state plays a minor role as direction in a vector space can be identified by a unit vector. In order to test our hypothesis, an original "experiment" has been performed. Normalised state vectors states has been considered, the set of $N$-dimensional vector states arranges over an $(N-1)$-dimensional sphere and factor models on the set of unit vector states have been calibrated.

An obvious effect of our spherical re-definition is a sort of compactification of the operational state clusters (Fig. 7). Our hypothesis on the characteristic variations of the covariance structure appears rather plausible. In fact points representing ordinary operational states and true damage situations form well-defined compact clusters.

Remarkably, as a result of our original approach, in this case the discrimination between true and false alerts becomes much more striking (compare Fig. 7 to Fig. 5). In this new situation the definition of the linear discriminant conditions appears even easier and precise with respect to the previous factor models.

The typical behaviour of the unit states of a power drive is that true damage states condense in a compact region inside the scatter-plot cluster of states. It is often easy to work-out a discriminant condition based on the affinity to that specific compact region. On Fig. 8 is shown the case of a TGB Gear.

Other advantage of the normalisation of the vector states is the elimination of the large spreading of false anomalous alerts far from the mean value of the normal operational regime. In this context LDA leads to precisely the same classification results, but remarkably QDA of the unit vector states of the "critical case" Hangar Ball Brg produces a slight improvement (compare Table 5 to Table 4).

In conclusion, our mathematical experiment led to interesting and in some cases unexpected, potentially useful results. The principal factor analysis on unit states gives further, often relevant, information on the anomalous behaviour of some mechanical components, and can be therefore integrated in a control procedure.

Table 5. Leave-one-out QDA re-classification of Hangar Ball Brg unit vector states

| real \ classified as | false alert | normal | true failure |
|---|---|---|---|
| false alert | 60 | 2 | 2 |
| normal operat. | 63 | 1432 | 67 |
| true damage | 2 | 33 | 136 |

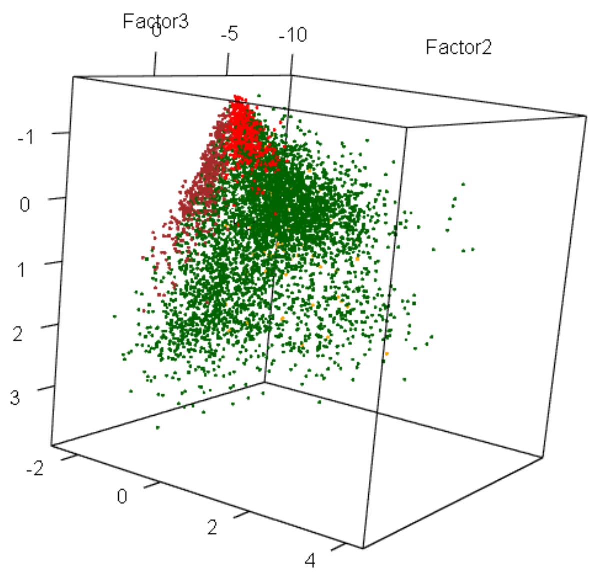


Figure 8. Bartlett type scores of unit states of a TGB Gear.

## 6. INTEGRATED CONTROL PROCESS, IMPLEMENTATION AND RESULTS

The statistical techniques tested over the available vector data set are based on different mathematical constructions and therefore provide different results. For this reason the above techniques have been combined in a software implementation of an integrated control process in the following way:

**1. Anomaly detection** by means of a self-learning control chart. A problem highlighted by the experts of AgustaWestland consists of the fact that the normal operational regime of some power drives on certain helicopters is characterised by very high values of the health indicators. Such values would be considered as anomalous if compared to other helicopters or to some a priori fixed threshold values. This ambiguity is completely removed by the self leaning individual calibration of the control chart. Any vector state judged in control contributes to the real time re-calibration of the control chart i.e. the control chart keeps learning.

**2. Anomaly classification** based on discriminant methods calibrated and validated over the entire fleet. A vector state judged as anomalous undergoes evaluation based on a set of distinguished discriminant techniques which can regard both the variance and the covariance structure of the calibration data sets (PCA, LDA, QDA, factor scores). A state classified as false alert does not generate an alert.

**3. Evaluation.** For different power drives, distinguished discriminant methods appear as more efficient. A pre-alert status is produced by a suitable combination of discriminant outputs. Such a combination is chosen in order to maximise the efficiency of the control system.

The integrated control process was then applied on a series of real cases contained in the historical database of AgustaWestland. In the case of the TGB gear and 2nd Stage Pin RH Brgs the integrated discriminant method judges a state as true failure i.e. generates a pre-alert if *each* discriminant method classifies it as a true failure. With this requirement only 3% of the measured states were miss-classified. In the most difficult case of Hangar Ball Brg a pre-alert is produced in 13% of the normal states, in 28% of the previous false alerts and in 65% of the true failure states. The current univariate version of the control system generates an alert if the values of the health indicators exceed the alarm thresholds in a fixed proportion (usually 2/3) in a number of consecutive measurements. In the integrated method these proportions can be deduced directly from these last results. For example, in the case of Hangar Ball Brg a suitable proportion appears 1/2.

An engineering software tool which implements both the control process and the calibration of the parameters of the control routine for each of the monitored power drives has been produced.

## 7. CONCLUSION

Our considerations have highlighted the advantages of our third-level multivariate approach. An efficient control process is based on an integration of several classification techniques. Even in those cases in which true failures and false alerts show misleading univariate profiles, multivariate techniques are able to distinguish them with very high level of statistical reliability.

The elimination of the deterministic influence of the environmental states of the helicopter, gives the possibility to compare rigorously states measured on different helicopters in different flight regimes. Once guaranteed this possibility, one can calibrate classification and discriminant models on historical data obtained from many helicopters and apply them in a future control process. When relevant new data are collected, the statistical models can be updated and improved by re-calibration on a larger and more detailed data set. Once a precise anomaly gets observed and diagnosed on one aircraft of the fleet, it can be diagnosed elsewhere by means of its specific multivariate health profile.

The results obtained by our research appear therefore as quite positive and encouraging.

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