Expected First Occurrence Time of Uncertain Future Events in One-Dimensional Linear Systems

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ABSTRACT

The rapid advancement of machine learning algorithms has significantly enhanced tools for monitoring system health, making data-driven approaches predominant in Prognostics and Health Management (PHM). In contrast, model-based approaches have seen modest progress, as they are often constrained by the need for prior knowledge of specific governing equations, limiting their applicability to a wide range of problems. Recently, rigorous theoretical foundations have been established to extend dynamical systems theory by incorporating prognosis of uncertain events. This article leverages this formal framework to introduce and demonstrate a fundamental mathematical result for one-dimensional linear dynamical systems. The presented theorem offers an analytical expression for approximating the expected time at which an event will first occur in the future. Unlike typical thresholds, this event is triggered by a hazard zone, defined as an uncertain event likelihood function over the system's state space. Applications of this theorem can be found in implementing real-time prognostic frameworks, where it is crucial to quickly estimate the magnitude of impending failures. Emphasis is placed on minimizing computational burden to facilitate prognostic decision-making.

NOMENCLATURE

k_p	Natural number denoting the present time.
k	Natural number denoting any time in the future so that $k > k_p$.
X_k	Random system state at time k.
$X_{k_p+1:k}$	Random system state trajectory between k_p and k: $X_{k_p+1:k} = \{X_{k_p+1}, X_{k_p+2}, \ldots, X_k\}$.
$\mathbb{X}_k$	Domain of X_k: $\mathbb{X}_k = \mathbb{R}$.
$\mathbb{X}_{k_p+1:k}$	Domain of $X_{k_p+1:k}$: $\mathbb{X}_{k_p+1:k} = \mathbb{R}^{k-k_p}$.
$\mathcal{E}$	Qualitative description of an event.
E_k	Binary random variable denoting the occurrence or not the event $\mathcal{E}$ at time k.
$\tau_{\mathcal{E}}$	Random variable depicting the first future occurrence time of the event $\mathcal{E}$.
$\mathbb{P}(\cdot)$	Probability mass function.
$p(\cdot)$	Probability density function.
$\mathbb{E}\{\cdot\}$	Expectation operator.
$\delta_x(\cdot)$	Dirac delta distribution located at x.
$\mathbb{1}_{\mathbb{A}}(\cdot)$	Indicator function of an arbitrary set $\mathbb{A}$.
$\mathcal{N}(\mu, \sigma^2)$	Gaussian distribution with mean μ and standard deviation σ.
$\Phi(\cdot)$	Cumulative distribution function of a standard Gaussian distribution.
$erf(\cdot)$	Error function.
$e_l(\cdot)$	Elementary symmetric polynomial of degree l.

1. INTRODUCTION

The event prognostic problem is undoubtedly the most challenging regarding system health monitoring for different reasons (Vachtsevanos & Zahiri, 2022). Frequently, the failure data available is rather scarce since, normally, and under a reliability engineering approach, preventive maintenance is carried out periodically to ensure a failure situation is rarely reached. On the other hand, the monitored systems tend to be complex and governed by sophisticated differential equations that are difficult to model due to the number of variables involved. Even with this, it is extremely difficult to find real-time numerical solutions whose information can be fused with empirical information obtained from sensor networks. Among these are the main reasons that have resulted in predominantly data-driven research efforts (Fink et al., 2020). This approach allows us to distance ourselves from complex equations governing degradation phenomena, since these algorithms can find underlying structures through the correlation of variables. This abstraction results in predictions such as correlations, which in practice can be effective; however, the causal relationships inherent to understanding physical phenomena are lost and, therefore, the ability to offer mathematical guarantees about the results (Zanga, Ozkirimli, & Stella, 2022). Even though the future is uncertain, it is desirable to have guarantees about the performance of prognostic algorithms.

Although Prognostics and Health Management (PHM) is a relatively young discipline, the problem of prognosticating the time at which a future event will occur predates PHM (Beichelt, 2001; Lee & Whitmore, 2003). Under the names *"First-Passage Time"* (FPT) (Siegert, 1951) or *"First Hitting Time"* (FHT) (Salminen, 1988), this problem has been addressed for years in mathematics, economics, physics, chemistry, among other disciplines (Redner, 2001). The problem is typically set in continuous time to model physical phenomena, and aims to find the probability distribution for when a variable reaches predetermined thresholds (Blake & Lindsey, 1973). It is assumed that the dynamic evolution of the variable is characterized by a specific stochastic process, such as a Poisson or Gamma process, or, alternatively, as a diffusion process via Brownian motion.

Recent work has introduced probability distributions for the prognostic problem in continuous and discrete-time settings. These formulations are agnostic regarding the stochastic process characterizing system dynamics and the event declaration criteria, moving beyond the simple threshold-crossing approach. This formulation, known as the Theory of Uncertain Event Prognosis (Acuña-Ureta, Orchard, & Wheeler, 2021), lays rigorous theoretical foundations for extending dynamical systems theory to prognostics of future events.

Building on this formulation, this article presents and proves a theorem that allows for determining the expected time of a future event with minimal computational effort. The analysis assumes a one-dimensional linear system, where the event is declared randomly, akin to tossing a coin (Bernoulli process), with the probability of event occurrence determined by the system's current state.

The expectation of future event times has been previously studied within the context of FPT and FHT problems, albeit for very specific stochastic processes (Klein, 1952; Gut, 1974; Robbins, 1976; Wickwire, 1979; Talkner, 1987; Dominé, 1995; Latouche & V., 1995; Gitterman, 2000; Kulkarni & Tzenova, 2002; Dybiec, Gudowska-Nowak, & Hänggi, 2006; Agliari, 2008; Zhang, Qi, Zhou, Xie, & Guan, 2009; Lefebvre, 2010; Bo & Lefebvre, 2011; Mattos, Mejía-Monasterio, Metzler, & Oshanin, 2012; Bénichou, Guérin, & Voituriez, 2015; Polizzi, Therien, & Beratan, 2016; Vanvinckenroye & Denoël, 2017). However, this article presents a fundamental result for the first time, in a scenario where the typical threshold is replaced by a more general notion of a hazard zone in which an event is declared randomly without crossing a specific threshold.

The article is structured as follows. Section 2 briefly illustrates and develops the intuition about a fundamental result that will lead to a theorem in Section 3. In this latter section, we will present the aforementioned theorem and proceed with its proof step-by-step. Finally, Section 4 presents the conclusions of this work.

2. OUTLINE OF A FUNDAMENTAL RESULT

Let us assume a one-dimensional linear system:

$$x_{k+1} = ax_k + \omega_k, \quad \omega_k \sim \mathcal{N}(0, {\sigma_\omega}^2), \tag{1}$$

where $x_k \in \mathbb{X}_k = \mathbb{R}$ is the system state at time $k \in \mathbb{N}$, that might describe the evolution over time of a health indicator, for example. Let $k_p \in \mathbb{N}$ denote the present time and x_{k_p} the current system state. Depending on the value of the parameter a, the state of the system will have an increasing ($a > 1$) or decreasing ($a < 1$) dynamic, as shown in Figs. 1 and 2, respectively. We establish that an event $\mathcal{E}$ occurs with some probability depending on the state of the system, which is known as *"hazard zone"* (Orchard & Vachtsevanos, 2009). In the context of PHM, the event could be defined qualitatively as

$$\mathcal{E} = \text{“System failure”}. \tag{2}$$

Without loss of generality, we can say that $a < 1$. We can define a binary stochastic process $\{E_k\}_{k>k_p}$ such that

$$\mathbb{P}\left(E_k = \mathcal{E} | x_k\right) = 1 - \Phi\left(\frac{x_k - \mu_h}{\sigma_h}\right), \tag{3}$$

where $\Phi(\cdot)$ is the cumulative distribution function of a standard Gaussian distribution. The binary random variable E_k indicates whether the event $\mathcal{E}$ occurs or not at each time k, $k > k_p$. The probabilities are given by Eq. (3). This can be understood as a particular way of defining a hazard zone. Note that the probability is never zero for any x_k that might be picked.

In summary, we have a linear system and a hazard zone defined through the statistics of a random variable E_k that describes the occurrence of an event $\mathcal{E}$, which could correspond to a catastrophic failure of the system. The prognostic prob-

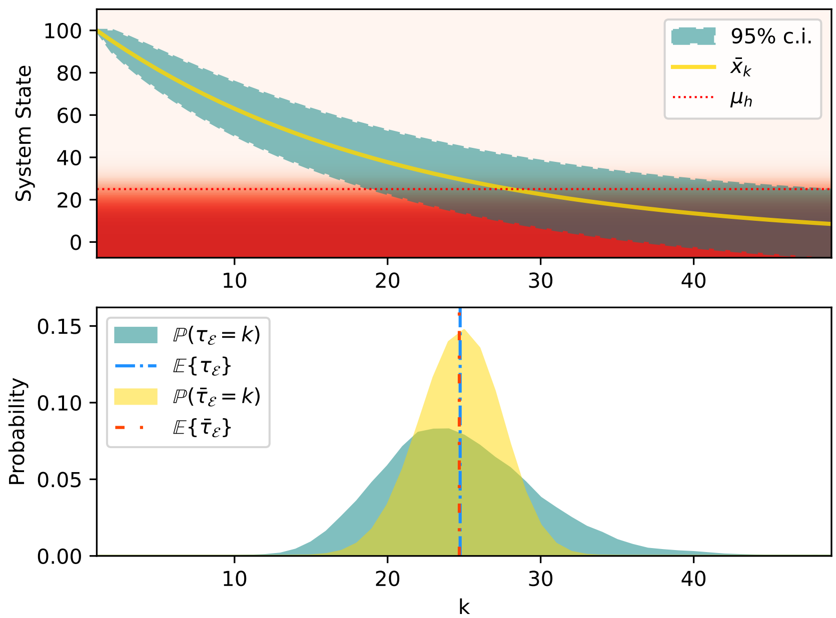

Figure 1. Illustration of the uncertain decreasing dynamics described by the Gaussian system state X_k of Eq. (1) when it is stable ($a < 1$). In the graph above it is shown its mean $\bar{x}_k$ (in yellow) with 95% confidence intervals (in green), and a *hazard zone* (as a red gradient) that triggers the occurrence of the event $\mathcal{E}$. The more intense the red color is for a value of x_k, the greater the probability that $\mathcal{E}$ occurs. The graph below shows the probability distribution of $\tau_{\mathcal{E}}$ and its expected value $\mathbb{E}\{\tau_{\mathcal{E}}\}$ using a vertical line. Two cases are simultaneously shown: when $\sigma_\omega > 0$ ($\mathbb{E}\{\tau_{\mathcal{E}}\}$, light blue) and when $\sigma_\omega = 0$ ($\mathbb{E}\{\bar{\tau}_{\mathcal{E}}\}$, orange). Regardless of the case, the expected value of the first occurrence time of $\mathcal{E}$ is shown to be apparently invariant. This is actually true for the *hazard zone* under study, and is stated as a theorem in Section 3.

lem consists of finding the time at which this event first occurs in the future, whose mathematical definition can be expressed as (Acuña-Ureta et al., 2021)

$$\tau_{\mathcal{E}}(k_p) = \inf \Big\{ k \in \mathbb{N} : \{k > k_p\} \wedge \{E_k = \mathcal{E}\} \Big\}. \quad (4)$$

Naturally, since the system has uncertain dynamics, this uncertainty results in $\{E_k\}_{k>k_p}$ being a binary stochastic process and, subsequently, $\tau_{\mathcal{E}}$ becoming a random variable.

A logical question could arise: What is the expected value of $\tau_{\mathcal{E}}$? Is there an analytical way to calculate it? The answer is "yes", the fundamental result presented in this article. Figure 1 is shown to facilitate understanding and generate intuition of a fundamental result framed as a theorem in Section 3.

In the upper graph of Figure 1, the vertical axis corresponds to the system state, and the horizontal axis is the time k. Since the system is linear with Gaussian additive process noise, then X_k is a Gaussian random variable, for each k, whose mean has been denoted as $\bar{x}_k$ and is plotted as a yellow line over time. The uncertainty associated with X_k is expressed with 95% confidence intervals shown through the area included in green. Finally, the red hue depicts the *hazard zone*, where red color intensity reflects the probability with which the event $\mathcal{E}$ occurs. With all this information, in the lower graph of Figure 1, the probability distribution of $\tau_{\mathcal{E}}$ is shown in green and is calculated according to the definition provided in Eq. (4). Note that its expected value is also shown there with a vertical line, which is $\mathbb{E}\{\tau_{\mathcal{E}}\}$.

Suppose that $\sigma_\omega = 0$. This would be equivalent to establishing that the dynamics of the system state is deterministic. This is, $X_k = \bar{x}_k$, for all $k > k_p$. The upper graph of Figure 1 shows this deterministic dynamic in yellow. Nonetheless, the event declaration criteria are still uncertain, so this deterministic trajectory of the system state still yields a probability distribution for the first time at which $\mathcal{E}$ occurs in the future. We denote this random variable as $\bar{\tau}_{\mathcal{E}}$, and its probability distribution is shown in the lower graph of Figure 1, in yellow color. Its expected value, therefore, is $\mathbb{E}\{\bar{\tau}_{\mathcal{E}}\}$ and is shown by a vertical line.

An analogous case for when the dynamics are increasing is shown in Figure 2.

Given what is evidenced by Figures 1 and 2, it is valid to raise the following questions:

- Could it be that $\mathbb{E}\{\tau_{\mathcal{E}}\} \approx \mathbb{E}\{\bar{\tau}_{\mathcal{E}}\}$?

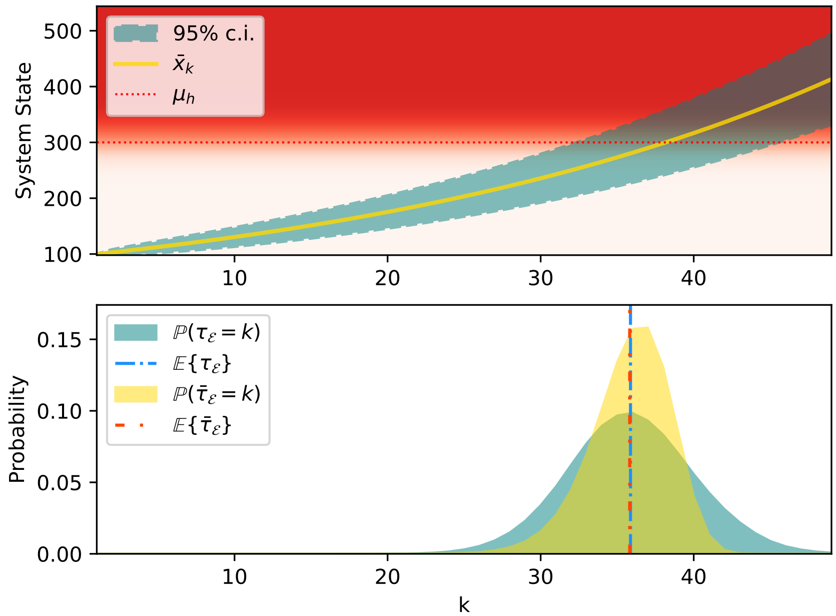

Figure 2. Illustration of the uncertain increasing dynamics described by the Gaussian system state X_k of Eq. (1) when it is unstable ($a > 1$). In the graph above it is shown its mean $\bar{x}_k$ (in yellow) with 95% confidence intervals (in green), and a *hazard zone* (as a red gradient) that triggers the occurrence of the event $\mathcal{E}$. The more intense the red color is for a value of x_k, the greater the probability that $\mathcal{E}$ occurs. The graph below shows the probability distribution of $\tau_\mathcal{E}$ and its expected value $\mathbb{E}\{\tau_\mathcal{E}\}$ using a vertical line. Two cases are simultaneously shown: when $\sigma_\omega > 0$ ($\mathbb{E}\{\tau_\mathcal{E}\}$, light blue) and when $\sigma_\omega = 0$ ($\mathbb{E}\{\bar{\tau}_\mathcal{E}\}$, orange). Regardless of the case, the expected value of the first occurrence time of $\mathcal{E}$ is shown to be apparently invariant. This is actually true for the *hazard zone* under study, and is stated as a theorem in Section 3.

- Will the above always be true in this type of system, regardless of the defined parameters?
- If all of the above is true. Will there be any advantage to calculating $\mathbb{E}\{\bar{\tau}_\mathcal{E}\}$ instead of $\mathbb{E}\{\tau_\mathcal{E}\}$?

The answers to all the questions posed are: “Yes”. The third question, in particular, is very interesting since it leads us to analytically calculate these expected values from a single deterministic trajectory of the system's state, which is its expected trajectory $\bar{x}_{k_p+1:k}$. In Section 3 that follows, this idea is formulated as a theorem immediately proven below. However, it is worth stopping first to analyze the implications of this fundamental theoretical result.

This theoretical result presents a novel strategy to reduce the computational load of failure prognostic algorithms for faster failure probability assessment. While it is relatively easy to compute the expected time of the event when the hazard zone is described by a threshold (just check the time at which the average state trajectory crosses the threshold), it is not straightforward to do this computation when the hazard zone is described in terms of an uncertain event likelihood, illustrated in Figures 1 and 2 as a gradient of red hues. It would be possible to simulate some state trajectories and average the times at which the events are recorded, in which case the accuracy of the estimate would be subject to the number of simulations: the more trajectories, the greater the accuracy of the average. All this effort would be just to know the average time of occurrence of the event, which could in fact occur even earlier than expected. It would be prudent to supplement this information in some way, but with the computational burden already assumed, it becomes difficult to obtain it through additional computing efforts, thus undermining the aspirations to make decisions in real-time. The theorem presented in Section 3 allows the computation to be done almost instantaneously by simulating a single deterministic state trajectory, and whose result is exact (with infinite precision).

From the point of view of a user that has to make decisions, the expected value for the time an event will occur could be of greater or lesser use. When the event is desirable, such as a particle reaching a certain energy level in physics, for example, knowing this expectation is fundamental. Given the speed of the phenomenon, an appropriate experimental setting would be needed to study this event in particles. If the standard deviation is also known, it would be even better. In such a case, we could quantify uncertainty with levels of precision. In contrast, the perspective is different when the event is undesirable, such as a catastrophic equipment failure. Here, the decision to make is when to repair or abort a mission. The expected value for the time a catastrophic failure will occur might not seem very useful at first glance since there is a probability of 0.5 that the event had occurred before. In this example of an undesirable event, the key is to characterize the uncertainty of the tail of the failure time distribution associated with early times, not late ones. However, even with these considerations, knowing the expected failure time means knowing the time window in which the decision must be made, and the further away it is from the expected time, the more conservative it will be. Having additional information could improve decision-making even more.

3. THEOREM FORMULATION

The theorem previously outlined in Section 2 is formulated and proven below. The code that generates Figures 1 and 2 is available[1] to replicate these results as well as to experiment with different parameters the veracity of this theorem.

Theorem (First Occurrence Time Expectation)**.** *Let us assume a one-dimensional linear system:*

$$x_{k+1} = a x_k + \omega_k,$$

where $k \in \mathbb{N}$ depicts discrete-time, $x_k \in \mathbb{X}_k = \mathbb{R}$ denotes the system state, $a \in \mathbb{R}_+$ is a model parameter, and ω_k is process noise such that $\mathbb{E}\{\omega_k\} = 0$. Considering that $k_p \in \mathbb{N}$ is the present time, assume $x_{k_p} > 0$. Let $\{E_k\}_{k>k_p}$ be a binary stochastic process describing the occurrence of an event $\mathcal{E}$, such that the probability with which it occurs at time k, $k > k_p$, depends on the value of x_k and is given by

$$\mathbb{P}\left(E_k = \mathcal{E} | x_k\right) = \begin{cases} 1 - \Phi\left(\frac{x_k - \mu_h}{\sigma_h}\right), & \textit{if } a < 1 \\ \Phi\left(\frac{x_k - \mu_h}{\sigma_h}\right) & , \textit{if } a > 1. \end{cases} \tag{5}$$

where $\mu_h \in \mathbb{R}$ and $\sigma_h \in \mathbb{R}_+$ are fixed parameters, and the function $\Phi(\cdot)$ corresponds to the cumulative distribution function of a standard Gaussian distribution. If the first occurrence time of $\mathcal{E}$ in the future exists and is denoted by the random variable $\tau_{\mathcal{E}}$, i.e. $\mathbb{P}\left(\tau_{\mathcal{E}} < +\infty\right) = 1$, then its expected value can be approximated as

$$\mathbb{E}\{\tau_{\mathcal{E}}\} \approx \sum_{k>k_p} k \mathbb{P}\left(E_k = \mathcal{E} | \bar{x}_k\right) \prod_{j=k_p+1}^{k-1} \left[1 - \mathbb{P}\left(E_j = \mathcal{E} | \bar{x}_j\right)\right], \tag{6}$$

where $\bar{x}_k = a^{k-k_p} x_{k_p}$.

Proof. According to the Theory of Uncertain Events Prognosis (Acuña-Ureta et al., 2021), the probability distribution of the future time of the first occurrence of the event $\mathcal{E}$, denoted as $\tau_{\mathcal{E}}$, is obtained using the joint probability distribution $p(x_{k_p+1:k})$ as

$$\mathbb{P}(\tau_{\mathcal{E}} = k) := \int_{\mathbb{X}_{k_p+1:k}} \mathbb{P}\left(E_k = \mathcal{E} | x_k\right) \prod_{j=k_p+1}^{k-1} \Big[1 \\ \ldots - \mathbb{P}\left(E_j = \mathcal{E} | x_j\right)\Big] p(x_{k_p+1:k}) dx_{k_p+1:k}. \tag{7}$$

Since $\mathbb{P}(\tau_{\mathcal{E}} = k)$ is calculated as an expected value, by the Law of Large Numbers it can be approximated with Monte Carlo simulations. If we draw N samples $x^{(i)}_{k_p+1:k} \sim p(x_{k_p+1:k})$ (equivalent to simulate N random trajectories), then we have the following weak convergence:

$$\frac{1}{N} \sum_{i=1}^{N} \delta_{x^{(i)}_{k_p+1:k}} \left(x_{k_p+1:k}\right) \underset{N \to +\infty}{\overset{w}{\to}} p(x_{k_p+1:k}), \tag{8}$$

where $\delta_{x^{(i)}_{k_p+1:k}} \left(x_{k_p+1:k}\right)$ is a Dirac delta located at $x^{(i)}_{k_p+1:k}$. Replacing $p(x_{k_p+1:k})$ by $\frac{1}{N}\sum_{i=1}^{N} \delta_{x^{(i)}_{k_p+1:k}}(x_{k_p+1:k})$ in Eq. (7) leads to an approximation of that expression. Therefore, we can associate the approximated expression with another random variable $\tau^N_{\mathcal{E}}$ such that

$$\mathbb{P}(\tau^N_{\mathcal{E}} = k) = \int_{\mathbb{X}_{k_p+1:k}} \mathbb{P}\left(E_k = \mathcal{E} | x_k\right) \prod_{j=k_p+1}^{k-1} \Big[1- \\ \ldots \mathbb{P}\left(E_j = \mathcal{E} | x_j\right)\Big] \left(\frac{1}{N} \sum_{i=1}^{N} \delta_{x^{(i)}_{k_p+1:k}} \left(x_{k_p+1:k}\right)\right) dx_{k_p+1:k} \tag{9}$$

$$= \frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{X}_{k_p+1:k}} \mathbb{P}\left(E_k = \mathcal{E} | x_k\right) \prod_{j=k_p+1}^{k-1} \Big[1- \\ \ldots \mathbb{P}\left(E_j = \mathcal{E} | x_j\right)\Big] \delta_{x^{(i)}_{k_p+1:k}} \left(x_{k_p+1:k}\right) dx_{k_p+1:k} \tag{10}$$

$$= \frac{1}{N} \sum_{i=1}^{N} \mathbb{P}\left(E_k = \mathcal{E} | x^{(i)}_k\right) \prod_{j=k_p+1}^{k-1} \left[1 - \mathbb{P}\left(E_j = \mathcal{E} | x^{(i)}_j\right)\right]. \tag{11}$$

By weak convergence, we have

$$\mathbb{P}(\tau_{\mathcal{E}} = k) = \lim_{N \to +\infty} \mathbb{P}(\tau^N_{\mathcal{E}} = k). \tag{12}$$

From here, it follows that

$$\Rightarrow \mathbb{E}\{\tau_{\mathcal{E}}\} = \lim_{N \to +\infty} \mathbb{E}\{\tau^N_{\mathcal{E}}\}. \tag{13}$$

Since we know that the previous equation is true, it is important to highlight that the strategy followed in the proof con-

[1]Code available at: https://github.com/SPAE-Research-Group/PHM24

sists of finding an alternative expression for $\mathbb{E}\{\tau_{\mathcal{E}}^N\}$. Then, we take the limit $N \to +\infty$.

Keeping the above in mind, we can express the i-th simulated future state at time k as

$$\begin{aligned} x_k^{(i)} &= a x_{k-1}^{(i)} + \omega_{k-1}^{(i)} && (14)\\ &= a\left(a x_{k-2}^{(i)} + \omega_{k-2}^{(i)}\right) + \omega_{k-1}^{(i)} && (15)\\ &= a^2 x_{k-2}^{(i)} + a\omega_{k-2}^{(i)} + \omega_{k-1}^{(i)} && (16)\\ &\vdots && (17)\\ &= a^{k-k_p} x_{k_p} + \sum_{n=1}^{k-k_p} a^{n-1}\omega_{k-n}^{(i)}, && (18) \end{aligned}$$

from which we can define

$$\begin{aligned} \bar{x}_k &= a^{k-k_p} x_{k_p} && (19)\\ \bar{\omega}_k^{(i)} &= \sum_{n=1}^{k-k_p} a^{n-1}\omega_{k-n}^{(i)}. && (20) \end{aligned}$$

Note that $\mathbb{E}\left\{\bar{\omega}_k^{(i)}\right\} = 0$. Indeed,

$$\begin{aligned} \mathbb{E}\left\{\bar{\omega}_k^{(i)}\right\} &= \mathbb{E}\left\{\sum_{n=1}^{k-k_p} a^{n-1}\omega_{k-n}^{(i)}\right\} && (21)\\ &= \sum_{n=1}^{k-k_p} a^{n-1}\mathbb{E}\left\{\omega_{k-n}^{(i)}\right\} && (22)\\ &= 0. && (23) \end{aligned}$$

Therefore, we can write

$$x_k^{(i)} = \bar{x}_k + \bar{\omega}_k^{(i)}. \quad (24)$$

If we are in the case that $a < 1$, then

$$\begin{aligned} \mathbb{P}\left(E_k = \mathcal{E} | x_k^{(i)}\right) &= 1 - \Phi\left(\frac{x_k^{(i)} - \mu_h}{\sigma_h}\right) && (25)\\ &= 1 - \Phi\left(\frac{\bar{x}_k - \mu_h + \bar{\omega}_k^{(i)}}{\sigma_h}\right) && (26)\\ &= 1 - \frac{1}{2}\left[1 + erf\left(\frac{\bar{x}_k - \mu_h + \bar{\omega}_k^{(i)}}{\sqrt{2}\sigma_h}\right)\right] && (27)\\ &= 1 - \frac{1}{2}\left[1 + \frac{2}{\sqrt{\pi}}\int_0^{\frac{\bar{x}_k - \mu_h + \bar{\omega}_k^{(i)}}{\sqrt{2}\sigma_h}} e^{-z^2} dz\right] && (28)\\ &= 1 - \frac{1}{2}\Bigg[1 + \frac{2}{\sqrt{\pi}}\Bigg(\int_0^{\frac{\bar{x}_k - \mu_h}{\sqrt{2}\sigma_h}} e^{-z^2} dz + \\ &\qquad \dots \int_{\frac{\bar{x}_k - \mu_h}{\sqrt{2}\sigma_h}}^{\frac{\bar{x}_k - \mu_h + \bar{\omega}_k^{(i)}}{\sqrt{2}\sigma_h}} e^{-z^2} dz\Bigg)\Bigg] && (29)\\ &= 1 - \Phi\left(\frac{\bar{x}_k - \mu_h}{\sigma_h}\right) - \frac{1}{\sqrt{\pi}}\int_{\frac{\bar{x}_k - \mu_h}{\sqrt{2}\sigma_h}}^{\frac{\bar{x}_k - \mu_h + \bar{\omega}_k^{(i)}}{\sqrt{2}\sigma_h}} e^{-z^2} dz. && (30) \end{aligned}$$

We can define

$$\begin{aligned} h_k(w) &:= \frac{1}{\sqrt{\pi}}\int_{\frac{\bar{x}_k - \mu_h}{\sqrt{2}\sigma_h}}^{\frac{\bar{x}_k - \mu_h + w}{\sqrt{2}\sigma_h}} e^{-z^2} dz && (31)\\ &= \frac{1}{\sqrt{2\pi}\sigma_h}\int_0^w e^{-\left(\frac{\bar{x}_k - \mu_h + \zeta}{\sqrt{2}\sigma_h}\right)^2} d\zeta. && (32) \end{aligned}$$

Provided $\mathbb{E}\left\{\bar{\omega}_k^{(i)}\right\} = 0$,

$$\Rightarrow \mathbb{E}\left\{h_k\left(\bar{\omega}_k^{(i)}\right)\right\} \approx 0. \quad (33)$$

This step is an important resource that we use at the end of the demonstration.

Eq. (25) can be re-expressed as

$$\mathbb{P}\left(E_k = \mathcal{E} | x_k^{(i)}\right) = \mathbb{P}\left(E_k = \mathcal{E} | \bar{x}_k\right) - h_k\left(\bar{\omega}_k^{(i)}\right). \quad (34)$$

Analogously, if we have $a > 1$, then

$$\begin{aligned} \mathbb{P}\left(E_k = \mathcal{E} | x_k^{(i)}\right) &= \Phi\left(\frac{x_k^{(i)} - \mu_h}{\sigma_h}\right) && (35)\\ &= \Phi\left(\frac{\bar{x}_k - \mu_h}{\sigma_h}\right) + \frac{1}{\sqrt{\pi}}\int_{\frac{\bar{x}_k - \mu_h}{\sqrt{2}\sigma_h}}^{\frac{\bar{x}_k - \mu_h + \bar{\omega}_k^{(i)}}{\sqrt{2}\sigma_h}} e^{-z^2} dz, && (36) \end{aligned}$$

and Eq. (35) can be re-expressed as

$$\mathbb{P}\left(E_k = \mathcal{E} | x_k^{(i)}\right) = \mathbb{P}\left(E_k = \mathcal{E} | \bar{x}_k\right) + h_k\left(\bar{\omega}_k^{(i)}\right). \quad (37)$$

Without loss of generality, from now on, we will assume in the proof that $a < 1$, since it only affects the sign that precedes $h_k\left(\bar{\omega}_k^{(i)}\right)$ in Eqs. (34) and Eq. (37), but the conclusion is indistinct from this sign since the terms associated with $h_k\left(\bar{\omega}_k^{(i)}\right)$ in both cases will be approximately zero in the end.

Hence, we can rewrite Eq. (11)

$$\begin{aligned} &\mathbb{P}(\tau_{\mathcal{E}}^N = k) \\ &= \frac{1}{N}\sum_{i=1}^{N}\mathbb{P}\left(E_k = \mathcal{E} | x_k^{(i)}\right)\prod_{j=k_p+1}^{k-1}\left[1 - \mathbb{P}\left(E_j = \mathcal{E} | x_j^{(i)}\right)\right] \end{aligned} \quad (38)$$

$$= \frac{1}{N}\sum_{i=1}^{N}\left(\mathbb{P}\left(E_k = \mathcal{E}|\bar{x}_k\right) - h_k\left(\bar{\omega}_k^{(i)}\right)\right)\prod_{j=k_p+1}^{k-1}\Big[1-$$
$$\ldots\left(\mathbb{P}\left(E_j = \mathcal{E}|\bar{x}_j\right) - h_j\left(\bar{\omega}_j^{(i)}\right)\right)\Big] \tag{39}$$
$$= \frac{1}{N}\sum_{i=1}^{N}\left(1 - 1 + \mathbb{P}\left(E_k = \mathcal{E}|\bar{x}_k\right) - h_k\left(\bar{\omega}_k^{(i)}\right)\right)$$
$$\ldots\cdot\prod_{j=k_p+1}^{k-1}\left[1 - \mathbb{P}\left(E_j = \mathcal{E}|\bar{x}_j\right) + h_j\left(\bar{\omega}_j^{(i)}\right)\right] \tag{40}$$
$$= \frac{1}{N}\sum_{i=1}^{N}\left(1 - \left[1 - \mathbb{P}\left(E_k = \mathcal{E}|\bar{x}_k\right) + h_k\left(\bar{\omega}_k^{(i)}\right)\right]\right)$$
$$\ldots\cdot\prod_{j=k_p+1}^{k-1}\left[1 - \mathbb{P}\left(E_j = \mathcal{E}|\bar{x}_j\right) + h_j\left(\bar{\omega}_j^{(i)}\right)\right] \tag{41}$$
$$= \frac{1}{N}\sum_{i=1}^{N}\left(\prod_{j=k_p+1}^{k-1}\left[1 - \mathbb{P}\left(E_j = \mathcal{E}|\bar{x}_j\right) + h_j\left(\bar{\omega}_j^{(i)}\right)\right]\right.$$
$$\left.\ldots - \prod_{j=k_p+1}^{k}\left[1 - \mathbb{P}\left(E_j = \mathcal{E}|\bar{x}_j\right) + h_j\left(\bar{\omega}_j^{(i)}\right)\right]\right) \tag{42}$$

If we define $r_j^{(i)} = -\frac{h_j\left(\bar{\omega}_j^{(i)}\right)}{1-\mathbb{P}(E_j=\mathcal{E}|\bar{x}_j)}$, we can apply Viète's Theorem (Viète, 1646) to yield a polynomial expansion in terms of *elementary symmetric polynomials* (Macdonald, 1995):

$$\prod_{j=k_p+1}^{k}\left(\lambda - r_j^{(i)}\right) = \lambda +$$
$$\ldots\sum_{l=1}^{k-k_p}(-1)^l\lambda^{k-k_p-l}e_l\left(r_{k_p+1}^{(i)}, r_{k_p+2}^{(i)}, \ldots, r_k^{(i)}\right), \tag{43}$$

where $e_l\left(r_{k_p+1}^{(i)}, r_{k_p+2}^{(i)}, \ldots, r_k^{(i)}\right)$ is an elementary symmetric polynomial of degree l. Adopting $\lambda = 1$, we can go back to Eq. (42) and make use of this property:

$$\mathbb{P}(\tau_{\mathcal{E}}^N = k)$$
$$= \frac{1}{N}\sum_{i=1}^{N}\left[\prod_{j=k_p+1}^{k-1}\left[1 - \mathbb{P}\left(E_j = \mathcal{E}|\bar{x}_j\right)\right]\left(1 - r_j^{(i)}\right) - \right.$$
$$\left.\ldots\prod_{j=k_p+1}^{k}\left[1 - \mathbb{P}\left(E_j = \mathcal{E}|\bar{x}_j\right)\right]\left(1 - r_j^{(i)}\right)\right] \tag{44}$$
$$= \frac{1}{N}\sum_{i=1}^{N}\left[\prod_{j=k_p+1}^{k-1}\left[1 - \mathbb{P}\left(E_j = \mathcal{E}|\bar{x}_j\right)\right]\left(1 + \right.\right.$$
$$\left.\ldots\sum_{l=1}^{k-k_p-1}(-1)^l e_l\left(r_{k_p+1}^{(i)}, r_{k_p+2}^{(i)}, \ldots, r_{k-1}^{(i)}\right)\right) -$$
$$\ldots\prod_{j=k_p+1}^{k}\left[1 - \mathbb{P}\left(E_j = \mathcal{E}|\bar{x}_j\right)\right]\left(1 + \right.$$
$$\left.\left.\ldots\sum_{l=1}^{k-k_p}(-1)^l e_l\left(r_{k_p+1}^{(i)}, r_{k_p+2}^{(i)}, \ldots, r_k^{(i)}\right)\right)\right] \tag{45}$$
$$= \mathbb{P}\left(E_k = \mathcal{E}|\bar{x}_k\right)\prod_{j=k_p+1}^{k-1}\left[1 - \mathbb{P}\left(E_j = \mathcal{E}|\bar{x}_j\right)\right]$$
$$\ldots + \prod_{j=k_p+1}^{k-1}\left[1 - \mathbb{P}\left(E_j = \mathcal{E}|\bar{x}_j\right)\right]\sum_{l=1}^{k-k_p-1}(-1)^l\left[\frac{1}{N}\right.$$
$$\left.\ldots\sum_{i=1}^{N}e_l\left(r_{k_p+1}^{(i)}, r_{k_p+2}^{(i)}, \ldots, r_{k-1}^{(i)}\right)\right] -$$
$$\ldots\prod_{j=k_p+1}^{k}\left[1 - \mathbb{P}\left(E_j = \mathcal{E}|\bar{x}_j\right)\right]\sum_{l=1}^{k-k_p}(-1)^l\left[\frac{1}{N}\right.$$
$$\left.\ldots\sum_{i=1}^{N}e_l\left(r_{k_p+1}^{(i)}, r_{k_p+2}^{(i)}, \ldots, r_k^{(i)}\right)\right]. \tag{46}$$

Let us define

$$\phi_k^N = \prod_{j=k_p+1}^{k}\left[1 - \mathbb{P}\left(E_j = \mathcal{E}|\bar{x}_j\right)\right]\sum_{l=1}^{k-k_p}(-1)^l\left[\frac{1}{N}\right.$$
$$\left.\ldots\sum_{i=1}^{N}e_l\left(r_{k_p+1}^{(i)}, r_{k_p+2}^{(i)}, \ldots, r_k^{(i)}\right)\right], \tag{47}$$

so that Eq. (46) can be briefly expressed as

$$\mathbb{P}(\tau_{\mathcal{E}}^N = k) = \mathbb{P}\left(E_k = \mathcal{E}|\bar{x}_k\right)\prod_{j=k_p+1}^{k-1}\left[1 - \mathbb{P}\left(E_j = \mathcal{E}|\bar{x}_j\right)\right]$$
$$\ldots + \phi_{k-1}^N - \phi_k^N. \tag{48}$$

Recall that

$$\mathbb{E}\{\tau_{\mathcal{E}}\} = \lim_{N\to+\infty}\mathbb{E}\{\tau_{\mathcal{E}}^N\} \tag{49}$$
$$= \lim_{N\to+\infty}\sum_{k>k_p}k\mathbb{P}(\tau_{\mathcal{E}}^N = k) \tag{50}$$
$$= \sum_{k>k_p}k\mathbb{P}\left(E_k = \mathcal{E}|\bar{x}_k\right)\prod_{j=k_p+1}^{k-1}\left[1 - \mathbb{P}\left(E_j = \mathcal{E}|\bar{x}_j\right)\right] +$$
$$\ldots\lim_{N\to+\infty}\sum_{k>k_p}k\left(\phi_{k-1}^N - \phi_k^N\right). \tag{51}$$

However,

$$\sum_{k>k_p}k\left(\phi_{k-1}^N - \phi_k^N\right) = \sum_{k>k_p}k\phi_{k-1}^N - \sum_{k>k_p}k\phi_k^N \tag{52}$$

$$= \sum_{k>k_p} (k-1+1)\phi_{k-1}^N - \sum_{k>k_p} k\phi_k^N \tag{53}$$

$$= \sum_{k>k_p} \phi_{k-1}^N + \sum_{k>k_p} (k-1)\phi_{k-1}^N - \sum_{k>k_p} k\phi_k^N \tag{54}$$

$$= \sum_{k>k_p} \phi_{k-1}^N + k_p \cancelto{0}{\phi_{k_p}^N}. \tag{55}$$

Thus,

$$\mathbb{E}\{\tau_{\mathcal{E}}\} = \sum_{k>k_p} k\mathbb{P}\left(E_k = \mathcal{E}|\bar{x}_k\right) \prod_{j=k_p+1}^{k-1} \left[1 - \mathbb{P}\left(E_j = \mathcal{E}|\bar{x}_j\right)\right] \dots + \lim_{N\to+\infty} \sum_{k>k_p} \phi_{k-1}^N. \tag{56}$$

The proof is completed if $\lim\limits_{N\to+\infty} \sum_{k>k_p} \phi_{k-1}^N \approx 0$. Indeed,

$$\lim_{N\to+\infty} \phi_{k-1}^N$$

$$= \lim_{N\to+\infty} \prod_{j=k_p+1}^{k} \left[1 - \mathbb{P}\left(E_j = \mathcal{E}|\bar{x}_j\right)\right] \sum_{l=1}^{k-k_p} (-1)^l \left[\frac{1}{N} \dots \sum_{i=1}^{N} e_l\left(r_{k_p+1}^{(i)}, r_{k_p+2}^{(i)}, \dots, r_k^{(i)}\right)\right] \tag{57}$$

$$= \prod_{j=k_p+1}^{k} \left[1 - \mathbb{P}\left(E_j = \mathcal{E}|\bar{x}_j\right)\right] \sum_{l=1}^{k-k_p} (-1)^l \left[\lim_{N\to+\infty} \frac{1}{N} \dots \sum_{i=1}^{N} e_l\left(r_{k_p+1}^{(i)}, r_{k_p+2}^{(i)}, \dots, r_k^{(i)}\right)\right]. \tag{58}$$

Notwithstanding,

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{i=1}^{N} e_l\left(r_{k_p+1}^{(i)}, r_{k_p+2}^{(i)}, \dots, r_k^{(i)}\right)$$

$$= \lim_{N\to+\infty} \frac{1}{N} \sum_{i=1}^{N} \sum_{k_p+1\le k_1\le k_2\le\ldots\le k_l\le k} \prod_{m=1}^{l} r_{k_m}^{(i)} \tag{59}$$

$$= \sum_{k_p+1\le k_1\le k_2\le\ldots\le k_l\le k} \left(\lim_{N\to+\infty} \frac{1}{N} \sum_{i=1}^{N} \prod_{m=1}^{l} r_{k_m}^{(i)}\right) \tag{60}$$

$$= \sum_{k_p+1\le k_1\le k_2\le\ldots\le k_l\le k} \left(-\lim_{N\to+\infty} \frac{1}{N} \sum_{i=1}^{N} \prod_{m=1}^{l} \dots \frac{h_{k_m}\left(\bar{\omega}_{k_m}^{(i)}\right)}{1 - \mathbb{P}\left(E_{k_m} = \mathcal{E}|\bar{x}_{k_m}\right)}\right) \tag{61}$$

$$= -\sum_{k_p+1\le\ldots\le k} \left(\frac{\lim\limits_{N\to+\infty} \frac{1}{N} \sum_{i=1}^{N} \prod_{m=1}^{l} h_{k_m}\left(\bar{\omega}_{k_m}^{(i)}\right)}{\prod_{m=1}^{l} \left[1 - \mathbb{P}\left(E_{k_m} = \mathcal{E}|\bar{x}_{k_m}\right)\right]}\right) \tag{62}$$

$$= -\sum_{k_p+1\le\ldots\le k} \left(\frac{\mathbb{E}\left\{\prod_{m=1}^{l} h_{k_m}\left(\bar{\omega}_{k_m}\right)\right\}}{\prod_{m=1}^{l} \left[1 - \mathbb{P}\left(E_{k_m} = \mathcal{E}|\bar{x}_{k_m}\right)\right]}\right) \tag{63}$$

$$\le -\sum_{k_p+1\le\ldots\le k} \left(\frac{\prod_{m=1}^{l} \mathbb{E}\left\{h_{k_m}\left(\bar{\omega}_{k_m}\right)\right\}}{\prod_{m=1}^{l} \left[1 - \mathbb{P}\left(E_{k_m} = \mathcal{E}|\bar{x}_{k_m}\right)\right]}\right) \tag{64}$$

$$\approx 0, \tag{65}$$

provided we know from Eq. (33) that

$$\left|\prod_{m=1}^{l} \mathbb{E}\left\{h_{k_m}\left(\bar{\omega}_{k_m}\right)\right\}\right| \ll \left|\mathbb{E}\left\{h_{k_q}\left(\bar{\omega}_{k_q}\right)\right\}\right| \approx 0, \tag{66}$$

for any $q \in \{1, 2, \dots m\}$.

Thus, we finally conclude that

$$\lim_{N\to+\infty} \phi_{k-1}^N \approx 0 \Rightarrow \lim_{N\to+\infty} \sum_{k>k_p} \phi_{k-1}^N \approx 0, \tag{67}$$

and, from Eq. (56), that

$$\mathbb{E}\{\tau_{\mathcal{E}}\} \approx \sum_{k>k_p} k\mathbb{P}\left(E_k = \mathcal{E}|\bar{x}_k\right) \prod_{j=k_p+1}^{k-1} \left[1 - \mathbb{P}\left(E_j = \mathcal{E}|\bar{x}_j\right)\right]. \tag{68}$$

□

4. CONCLUSION

In this article, a new theorem for one-dimensional linear systems has been stated and proved within the Theory of Uncertain Event Prognosis framework. The theorem consists of establishing analytically, with a simple calculation, how to compute an estimation for the expectation of the time at which a future event will occur when its declaration is not based on simple thresholds but on a likelihood function covering the entire state-space, thus denoting what is known as a *"hazard zone"*. That is, for each value that the system's state takes in time, there is a probability that the event will occur. This, of course, induces a probability distribution over the time the event will occur for the first time, and the theorem precisely determines the expected value of that distribution. Despite the simplicity assumed in the studied dynamical system, this fundamental result constitutes an important advance in the area of event prognostics since it is not a method but an absolute truth within a logical-mathematical framework that can be verified (codes are provided) and extended to more complex systems.

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ACKNOWLEDGMENT

This work has been partially supported by Office of Naval Research Global (ONRG) Grant Nr. N62909-22-1-2056, FONDECYT Chile Grant Nr. 11231148, FONDECYT Chile Grant Nr. 1210031, and the Advanced Center for Electrical and Electronic Engineering, AC3E, Basal Project FB0008, ANID.

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