Bayesian fatigue damage and reliability analysis using Laplace approximation and inverse reliability method

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ABSTRACT

This paper presents an efficient analytical Bayesian method for reliability and system response estimate and update. The method includes additional data such as measurements to reduce estimation uncertainties. Laplace approximation is proposed to evaluate Bayesian posterior distributions analytically. An efficient algorithm based on inverse first-order reliability method is developed to evaluate system responses given a reliability level. Since the proposed method involves no simulations such as Monte Carlo or Markov chain Monte Carlo simulations, the overall computational efficiency improves significantly, particularly for problems with complicated performance functions. A numerical example and a practical fatigue crack propagation problem with experimental data are presented for methodology demonstration. The accuracy and computational efficiency of the proposed method is compared with simulation-based methods.

1. INTRODUCTION

Efficient inference on reliability and responses of engineering systems has drawn attention to the prognostics and health management society due to the increasing complexity of those systems (Melchers, 1999; Brauer & Brauer, 2009). For high reliability demanding systems such as aircraft and nuclear facilities, time-dependent reliability degradation and performance prognostics must be quantified to prevent potential system failures. Reliable predictions of system reliability and system responses are usually required for decision-making in a time and computational resource constrained situation. The basic idea of time-independent component reliability analysis involves computation of a multi-dimensional integral over the failure domain of the performance function (Madsen, Krenk, & Lind, 1986; Ditlevsen & Madsen, 1996; Rackwitz, 2001). For many practical problems with high-dimensional parameters, the exact evaluation of this integral is either analytically intractable or computationally infeasible with a given time constraint. Analytical approximations and numerical simulations are two major computational methods to solve this problem (Rebba & Mahadevan, 2008).

The simulation-based method includes direct Monte Carlo (MC) (Kalos & Whitlock, 2008), Importance Sampling (IS)(Gelman & Meng, 1998; Liu, 1996), and other MC simulations with different sampling techniques. Analytical approximation methods, such as first- and second- order reliability methods (FORM/SORM) have been developed to estimate the reliability without large numbers of MC simulations. FORM and SORM computations are based on linear (first-order) and quadratic (second-order) approximations of the limit-state surface at the *most probable point* (MPP)(Madsen et al., 1986; Ditlevsen & Madsen, 1996). Under the condition that the limit-state surface at the MPP is close to its linear or quadratic approximation and that no multiple MPPs exist in the limit-state surface, FORM/SORM are sufficiently accurate for engineering purposes(Bucher et al., 1990; Cai & Elishakoff, 1994; Zhang & Mahadevan, 2001; Zhao & Ono, 1999). If the final objective is to calculate the system response given a reliability index, the inverse reliability method can be used. The most well-known approach is inverse FORM method proposed in (Der Kiureghian, Yan, & Chun-Ching, 1994; Der Kiureghian & Dakessian, 1998; Li & Foschi, 1998). Several studies for static failure using the inverse FORM method have been reported in the literature. (Du, Sudjianto, & Chen, 2004) proposed an inverse reliability strategy and applied it to the integrated robust and reliability design of a vehicle combustion engine piston. (Saranyasoontorn & Manuel, 2004) developed an inverse reliability procedure for wind turbine components. (Lee, Choi, Du, & Gorsich, 2008) used the inverse reliability analysis for reliability-based design optimization of nonlinear multidimensional systems. (Cheng, Zhang, Cai, & Xiao, 2007) presented an artificial neural network based inverse FORM method for solving problems with complex and implicit performance functions. (Xiang & Liu, 2011) applied the inverse FORM method to time-dependent fatigue life predictions.

Conventional forward and inverse reliability analysis is based on the existing knowledge about the system (e.g., underlying physics, distributions of input variables). Time-dependent reliability degradation and system response changing are not reflected. For many practical engineering problems, usage monitoring or inspection data are usually available at a regular time interval either via structural health monitoring system or non-destructive inspections. The new information can be used to update the initial estimate of sys-

tem reliability and responses. The critical issue is how to incorporate the existing knowledge and new information into the estimation. Many methodologies have been proposed to handle reliability updating problems. Bayesian updating is the most common approach to incorporate these additional data. By continuous Bayesian updating, all the variables of interest are updated and the inference uncertainty can be significantly reduced, provided the additional data are relevant to the problem and they are informative. (Hong, 1997) presented the idea of reliability updating using inspection data. (Papadimitriou, Beck, & Katafygiotis, 2001) reported a reliability updating procedure using structural testing data. (Graves, Hamada, Klamann, Koehler, & Martz, 2008) applied the Bayesian network for reliability updating. (Wang, Rabiei, Hurtado, Modarres, & Hoffman, 2009) used Bayesian reliability updating for aging airframe. A similar updating approach using Maximum relative Entropy principles has also been proposed in (Guan, Jha, & Liu, 2009). In those studies, MCMC simulations have been extensively used to draw samples from posterior distributions. The Convergence Theorem ensures the resulting Markov chain converges to the target distribution (Gilks, Richardson, & Spiegelhalter, 1996) and it becomes almost a standard approach for Bayesian analysis with complex models. For practical problems with complicated performance functions, simulations are time-consuming and efficient computations are critical for time constrained reliability evaluation and system response prognostics. Some of the existing analytical methods includes variational methods (Ghahramani & Beal, 2000) and expectation maximization methods (Moon, 1996). Those methods usually focus on the approximation of distributions and does not provide a systematical procedure for inverse reliability problems. In structural health management settings, simulation-based method may be infeasible because updating is frequently performed upon the arrival of sensor data. All these application require efficient and accurate computations. However, very few studies are available on the investigation of complete analytical updating and estimation procedure without using simulations.

The objective of the proposed study is to develop an efficient analytical method for system reliability and response updating without using simulations. Three computational components evolved in this approach are Bayesian updating, reliability estimation, and system response estimation given a reliability or a confidence level. For Bayesian updating, Laplace method is proposed to obtain an analytical representation of the Bayesian posterior distribution and avoid MCMC simulations. Once the analytical posterior distribution is obtained, FORM method can be applied to update system reliability or probability of failure. In addition, predictions of system response associated with a reliability or a confidence level can also be updated using inverse FORM method to avoid MC simulations.

The paper is organized as follows. First, a general Bayesian posterior model for uncertain variables is formulated. Relevant information such as response measures and usage monitoring data are used for updating. Then an analytical approximation to the posterior distribution is derived based on Laplace method. Next, FORM method is introduced to estimate system reliability levels and a simplified algorithm based on inverse FORM method is formulated to calculate system response given a reliability level or a confidence level. Following that, numerical and application examples are presented to demonstrate the proposed method. The efficiency and accuracy of the proposed method are compared with simulation results.

2. PROBABILISTIC MODELING AND LAPLACE APPROXIMATION

In this section, a generic posterior model for uncertain parameters is formulated using Bayes' theorem to incorporate additional data such as measurements. Uncertainties from model parameters, measurement, and model independent variables are systematically included. To avoid MCMC simulations as in classical Bayesian applications, Laplace approximation is derived to obtain an analytical representation of the posterior distribution. The updated reliability and system responses can readily be evaluated using this posterior approximation.

2.1 Bayesian modeling for uncertain parameters

Consider a generic parameterized model $\mathcal{M}(y; x)$ describing an observable event d, where x is an uncertain model parameter vector and y is model independent variable. If the model is perfect, one obtains $\mathcal{M}(y; x) = d$. In reality, such a perfect model is rarely available due to uncertainties such as the simplification of the actual complex physical mechanisms, statistical error in obtaining the parameter x, and the measurement error in d. Using probability distributions to describe those uncertainties is a common practice.

Given the prior probability distribution of x, $p(x|\mathcal{M})$, and the known relationship (conditional probability distribution or likelihood function) between d and x, $p(d|x, \mathcal{M})$, the posterior probability distribution $p(x|d, \mathcal{M})$ is expressed using Bayes' theorem as

$$p(x|d, \mathcal{M}) = p(x|\mathcal{M})p(d|x, \mathcal{M})\frac{1}{\mathcal{Z}} \propto p(x|\mathcal{M})p(d|x, \mathcal{M}), \quad (1)$$

where $\mathcal{Z} = \int_X p(x|\mathcal{M})p(d|x, \mathcal{M})\mathrm{d}x$ is the normalizing constant.

The model $\mathcal{M}$ is assumed to be the only feasible model and $\mathcal{M}$ is omitted hereafter for simplicity. Let m be the model prediction and e the error term (for example, the measurement error of d). The variable d reads

$$d = m + e. \quad (2)$$

The probability distribution for m is represented by the function $p(m|x) = f_M(m)$ and the probability distribution for e is by the function $p(e|x) = f_E(e)$. The conditional probability distribution of $p(d|x)$ can be obtained by marginalizing the joint probability distribution of $p(d, m, e|x)$ as follows:

$$p(d|x) = \int_M \int_E p(m|x)p(e|x)p(d, m, e|x)\mathrm{d}e\mathrm{d}m. \quad (3)$$

Because $d = m + e$,

$$p(d, z, e|x) = \delta(d - m - e). \quad (4)$$

Substitute Eq. (4) into Eq. (3) to obtain

$$p(d|x) = \int_M f_M(m)f_E(d - m)\mathrm{d}m. \quad (5)$$

Next, terms $f_M(m)$ and $f_E(e)$ need to be determined. Consider a general case where the model prediction m has a statistical noise component $\epsilon \in \mathcal{E}$ with a distribution function $p(\epsilon|x) = f_{\mathcal{E}}(\epsilon)$ due to the modeling error $m = \mathcal{M}(y; x) + \epsilon$. Equation (2) is revised as

$$d = \mathcal{M}(y; x) + \epsilon + e. \quad (6)$$

Marginalizing $p(m|\epsilon, \theta) = \delta(m - \mathcal{M}(y; x) - \epsilon)$ over ϵ to obtain

$$f_M(m) = \int_{\mathcal{E}} p(\epsilon|x)p(m|x, \theta)\mathrm{d}\epsilon = f_{\mathcal{E}}(m - \mathcal{M}(y; x)). \quad (7)$$

2

For the purpose of illustration, ϵ and e are assumed to be two independent Gaussian variables with standard deviations of σ_ϵ and σ_e, respectively. This assumption is usually made when no other information about the uncertain variables is available (Gregory, 2005). Equation (5) is the convolution of two Gaussians and it can be further reduced to another Gaussian distribution as

$$p(d|x) = \frac{1}{\sqrt{2\pi(\sigma_\epsilon^2 + \sigma_e^2)}} exp \left[\frac{-(d - \mathcal{M}(y;x))^2}{2(\sigma_\epsilon^2 + \sigma_e^2)} \right]. \quad (8)$$

Substituting Eq. (3) into Eq. (1) yields the posterior probability distribution of the uncertain parameter x incorporating the observable event d. The reliability or system state variables can readily be updated with Eq. (1). For problems with high dimensional parameters, the evaluation of Eq. (1) is rather difficult because the exact normalizing constant $\mathcal{Z}$, which is a multi-dimensional integral, is either analytically intractable or computationally expensive. Instead of evaluating this equation directly, the most common approach is to draw samples from it using MCMC simulations. For applications where performance functions are computationally expensive to evaluate, this approach is time-consuming and hence not suitable for online updating and prognostics. To improve the overall computational efficiency, Laplace method is proposed to approximate the non-normalized Bayesian posterior distribution of $p(x|d)$. The derivation of Laplace approximation is presented below.

2.2 Laplace approximation for Bayesian posterior distributions

Consider the above non-normalized multivariate distribution $p(x|d)$ in Eq. (1) and its natural logarithm $\ln p(x|d)$. Expanding $\ln p(x|d)$ using Tylor series around an arbitrary point x^* yields

$$\ln p(x|d) = \ln p(x^*|d) + (x - x^*)^T \nabla \ln p(x^*|d) +$$
$$\frac{1}{2!}(x - x^*)^T \left[\nabla^2 \ln p(x^*|d) \right] (x - x^*) + \quad (9)$$
$$\mathrm{O}((x - x^*)^3),$$

where $\nabla \ln p(x^*|d)$ is the gradient of $\ln p(x|d)$ evaluated at x^*, $\nabla^2 \ln p(x^*|d)$ is the Hessian matrix evaluated at x^*, and $\mathrm{O}(\cdot)$ are higher-order terms. Assume that the higher-order terms are negligible in computation with respect to the other terms. We obtain

$$\ln p(x|d) \approx \ln p(x^*|d) + \underbrace{(x - x^*)^T \nabla \ln p(x^*|d)}_{(*)} +$$
$$\frac{1}{2!}(x - x^*)^T \left[\nabla^2 \ln p(x^*|d) \right] (x - x^*). \quad (10)$$

The term $(*)$ is zero at local maxima (denoted as x_0) of the distribution since $\nabla \ln p(x_0|d) = 0$. Therefore, if we choose to expand $\ln p(x|d)$ around x_0, we can eliminate term $(*)$ in Eq. (10) to obtain

$$\ln p(x|d) \approx \ln p(x_0|d) + \tfrac{1}{2}(x - x_0)^T \left[\nabla^2 \ln p(x_0|d) \right] (x - x_0). \quad (11)$$

Exponentiating $\ln p(x|d)$ of Eq. (11) yields

$$e^{\ln p(x|d)} \approx p(x_0|d) \exp \left\{ -\tfrac{1}{2}(x - x_0)^T [-\nabla^2 \ln p(x_0|d)](x - x_0) \right\}. \quad (12)$$

The last term of Eq. (12) resembles remarkably a multivariate Gaussian distribution with a mean vector of x_0 and a covariance matrix $\Sigma = \left[-\nabla^2 \ln p(x_0|d) \right]^{-1}$. The normalizing constant is

$$\mathcal{Z} = \int_X e^{\ln p(x|d)} \mathrm{d}x \approx p(x_0|d) \sqrt{(2\pi)^n |\Sigma|}, \quad (13)$$

where $\Sigma = \left[-\nabla^2 \ln p(x_0|d) \right]^{-1}$, n is the dimension of the variable x, and $|\Sigma|$ is the determinant of Σ.

The non-normalized Bayesian posterior distribution $p(x|d)$ is now approximated as

$$p(x|d) \approx \frac{1}{\sqrt{(2\pi)^n |\Sigma|}} \exp \left\{ -\tfrac{1}{2}(x - x_0)^T [\Sigma^{-1}](x - x_0) \right\}, \quad (14)$$

which is a multivariate Gaussian distribution with a mean vector of x_0 and a covariance matrix Σ. To compute x_0 and Σ, the first step is to find the local maxima of $\ln p(x|d)$ and evaluate the Hessian of $\ln p(x|d)$ at the local maxima. Numerical root-finding algorithms can be used to find local maxima, such as Gauss-Newton algorithm (Dennis Jr, Gay, & Walsh, 1981), Levenberg-Marquardt algorithm (More, 1978), trust-region dogleg algorithm (Powell, 1970), and so on. Laplace method can yield accurate results given the target distribution is approximately Gaussian distributed, which is quite common for practical problems (Gregory, 2005).

With the analytical representation of the posterior distribution $p(x|d)$, the updated reliability index can be calculated using the FORM method. In addition, updated system response predictions associated with a reliability index or a confidence level can also be calculated using inverse FORM method. For the sake of completeness, the basic concept of the FORM and inverse FORM methods are introduced briefly.

3. FORM AND INVERSE FORM METHODS

The time-invariant reliability analysis entails computation of a multi-dimensional integral over the failure domain of the performance function.

$$P_F \equiv P[g(x) < 0] = \int_{g(x) < 0} f_X(x) \mathrm{d}x, \quad (15)$$

where $x \in \mathcal{R}^n$ is a real-valued n-dimensional uncertain variable, $g(x)$ is the performance function, such that $g(x) < 0$ represents the failure domain, P_F is the probability of failure, and $f_X(x)$ is the joint probability distribution of x. The surface $g(x) = 0$ is usually called limit-state surface. In FORM/SORM methods, the uncertain variable is usually transformed from the standard probability space to the standard Gaussian space, also referred to as *reduced variable space*. Denote the transformed performance function as $g(z)$, where $z \in \mathcal{R}^n$ is an n-dimensional standard Gaussian variable, also called *reduced variable*. The distance between the closest point (most probable point (MPP), labeled as MPP in Figure 1) on the limit-state surface $g(z) = 0$ to the origin in the reduced variable space is the Hasofer-Lind reliability index (Madsen et al., 1986), denoted as β_{HL} in Figure 1. MPP is also known as the *design point*.
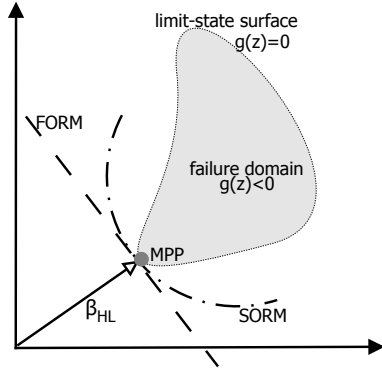
Figure 1: Linear (FORM) and quadratic (SORM) approximations of the performance function at MPP on the limit-state surface.

Reliability analysis entails the computation of β_{HL} and the design point, which is a standard constrained optimization problem defined as

$$\text{minimize: } ||z|| \quad \text{subject to } g(z) = 0, \qquad (16)$$

where $||z||$ denotes the distance between the point z and the origin in the reduced variable space.

The design point is generally not known a priori, hence an iterative process is required to find the design point z^* in the reduced variable space such that $\beta_{HL} \equiv ||z^*||$ corresponds to the shortest distance between z^* and the origin of the reduced variable space. Because reduced variables are based on the mean and standard deviation of a normal distribution, the non-normal variables must be transferred to its equivalent normal distribution. Rackwitz-Fiessler (Madsen, 1977) procedure is usually adopted for this purpose. The idea requires the cumulative density function (CDF) and the probability density function (PDF) of the target distribution be equal to a normal CDF and PDF at the value of variable x on the limit-state surface. This procedure finds the mean μ_{eq} and standard deviation σ_{eq} of the equivalent normal distribution and thus the variable x can be reduced to a standard Gaussian variable $z = (x - \mu_{eq})/\sigma_{eq}$. Several algorithms are available to locates the design point z^*, for example the Hasofer & Lind - Rackwitz & Fiessler (HL-RF) algorithm (Hasofer & Lind, 1974; Rackwitz & Flessler, 1978). With an initial guess of z_0 on the limit-state surface, the basic procedure computes the new location for z^* iteratively according to

$$z_{k+1} = \frac{1}{|\nabla g(z_k)|^2} \left[\nabla g(z_k) z_k - z_k \right] \nabla g(z_k)^T. \qquad (17)$$

A reasonable guess can be fixing the first $n-1$ components of z_0 to its distribution means and solving for the last component on the limit-state surface. The iterative procedure terminates based on some criteria such as $|\beta_{k+1} - \beta_k| < \epsilon_\beta$, where ϵ_β is a small control parameter assigned by users. Usually a value of $\epsilon_\beta = 10^{-4}$ to 10^{-3} yields accurate results for β_{HL} and the design point(Cheng et al., 2007).

After finding the design point and β_{HL} by solving Eq. (16) using the iterative formula of Eq. (17), FORM or SORM can approximate the probability of failure using a linear or quadratic approximation of the performance function, respectively. Both of them are based on Taylor series expansion of the performance function around the design point truncated

to linear and quadratic terms. For example, using FORM method yields the probability of failure as

$$P_F^{\text{FORM}} \cong \Phi(-\beta_{HL}), \qquad (18)$$

where Φ is the standard Gaussian CDF. The precision of this approximation depends on the non-linearity of the limit-state surface. Experience shows that FORM method yields accurate results for general engineering purposes (Cheng et al., 2007). FORM is a widely used computational model in reliability index approach (RIA) for reliability-based design optimization (RBDO) since it finds the reliability index β_{HL}. The advantage of RIA is that the probability of failure is forwardly calculated for a given design. However, inverse reliability analysis in performance measure approach (PMA) is known to be more robust and informative than the reliability analysis in RIA (Tu, Choi, & Park, 1999; Youn, Choi, & Du, 2005). The idea of inverse reliability analysis in PMA is to investigate whether a given design satisfies the probabilistic constraint with a target reliability index β_t. The inverse reliability analysis can also be expressed as an optimization problem such that

$$\text{minimize: } g(z) \quad \text{subject to } ||z|| = \beta_t. \qquad (19)$$

In inverse reliability analysis, among the different values of performance function $g(z)$ taking on z that pass through the β_t curve in the reduced variable space, the one z^* that minimizes the performance function is sought. Figure 2 illustrates the inverse reliability analysis. The point z^* is also called MPP and the corresponding minimal value of $g(z^*)$ is called probabilistic performance measure (PPM). Both reliability analysis and inverse reliability analysis search for MPPs. The difference is that the former search for the MPP on the limit-state surface $g(z) = 0$ while the latter search for MPP on the β_t curve. Based on the idea of inverse FORM procedure proposed in (Der Kiureghian et al., 1994), an efficient and simplified iterative formula in the reduced variable space is formulated as:

$$z_{k+1} = z_k + \lambda \left[-\beta_t \frac{\nabla g(z_k)}{|\nabla g(z_k)|} - z_k \right], \qquad (20)$$

where ∇ is the gradient vector with respect to z and λ is the step size at the kth iteration (a small constant is used in this formula instead of an adaptive value). The initial value z_0 is usually assigned to the distribution mean value. The iterative procedure proceeds until a convergence is achieved, i.e., when

$$\frac{|z_{k+1} - z_k|}{|z_{k+1}|} \leq \varepsilon, \qquad (21)$$

where ε is a small quantity assigned by the user. For practical problems, $\epsilon_t = 10^{-4}$ to 10^{-3} usually yields satisfactory estimates (Cheng et al., 2007). Based on the iterative formula, an algorithm locating MPP in inverse reliability problems is given as Algorithm 1.

Algorithm 1 Inverse FORM algorithm solving MPP given a target reliability index β_t

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1: Initiate z_0 and λ, set $k = 0$
2: **repeat**
3:     calculate z_{k+1} according to Eq. (20)
4:     calculate $d = \frac{|z_{k+1} - z_k|}{|z_{k+1}|}$
5:     $k \leftarrow k + 1$
6: **until** $d \leq \epsilon_t$

For a given confidence level, the reliability indexes associated with that level should be first calculated using inverse Gaussian CDF then the MPPs associated with these indexes can be calculated using Algorithm 1. System responses are readily evaluated with these MPPs.
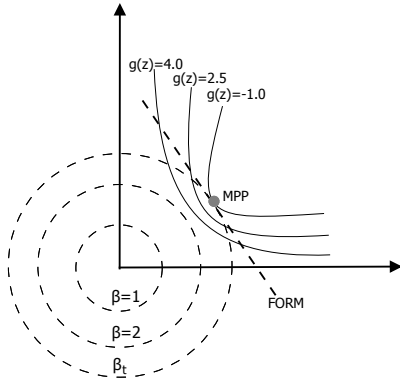

Figure 2: Inverse reliability analysis and MPP for target probability of failure of β_t and the linear approximation of the performance function at MPP labeled as FORM. Values of $g(z)$ are for illustration purposes only.

Both iterative formulae in Eq. (17) and Eq. (20) implicitly assumes that the components of z are uncorrelated. For correlated component variables in z, the correlated components need to be transformed into uncorrelated components via the orthogonal transformation of $z' = L^{-1}(z^T)$, where L is the lower triangular matrix obtained by Cholesky factorization of the correlation matrix R such that $LL' = R$, where L' is the transpose of L.

The overall computational procedure according to the proposed method is summarized as follows:

1. Formulate Bayesian posterior distributions according to Eq. (8).

2. Compute the posterior approximation according to Eq. (14).

3. Reliability or probability of failure estimation is calculated using iterative formula of Eq. (17) and Eq. (18).

4. To estimate system responses associated with a reliability level or confidence level, calculate MPPs using Algorithm 1 and then calculate system responses with the obtained MPPs.

Prior estimations are evaluated according to Steps 3 and 4 using prior distributions. To illustrate the proposed method, several examples are presented in the next section.

4. EXAMPLES

A numerical example is given first to illustrate the overall procedure, and a practical fatigue crack propagation problem with experimental data and a beam example with finite element analysis data are demonstrated. Comparisons with traditional simulation-based methods are made to investigate the accuracy and computational efficiency of the proposed method.

4.1 A numerical example with two uncertain variables

Consider a performance function $f(x, y) = x + y$ describing an observable event $z = f(x, y) + \epsilon$, where x and y are two uncertain variables and ϵ is an Gaussian error term with zero mean and a standard deviation of $\sigma_\epsilon = 0.5$. Variable x is normally distributed with a mean of $\mu_x = 2$ and a standard deviation of $\sigma_x = 0.5$ and variable y is also normally distributed with a mean $\mu_y = 5$ and a standard deviation $\sigma_y = 1.5$. Variables x and y are correlated with a correlation coefficient of $\rho_{xy} = -0.5$. The covariance matrix is $\Sigma_{xy} = \begin{bmatrix} \sigma_x^2 & \sigma_x \sigma_y \rho_{xy} \\ \sigma_x \sigma_y \rho_{xy} & \sigma_y^2 \end{bmatrix}$. $f(x, y) > 9$ is defined as failure event and the limit-state surface is $f(x, y) - 9 = 0$. The likelihood function can be expressed according to Eq. (8) as

$$p(z|x) = \frac{1}{\sqrt{2\pi}\sigma_\epsilon}\exp\left\{-\frac{1}{2}\left[\frac{z - f(x, y)}{\sigma_\epsilon}\right]^2\right\}. \quad (22)$$

Assume that the evidence of $z = 8$ is observed. The posterior distribution that encodes this information is formulated according to Eq. (1) as,

$$p(x|z) = \frac{1}{\sqrt{2\pi |\Sigma_{xy}|}}\exp\left\{-\frac{1}{2}\begin{pmatrix} x - \mu_x \\ y - \mu_y \end{pmatrix}^T [\Sigma_{xy}]^{-1}\begin{pmatrix} x - \mu_x \\ y - \mu_y \end{pmatrix}\right\} \times$$
$$\frac{1}{\sqrt{2\pi}\sigma_\epsilon}\exp\left\{-\frac{1}{2}\left[\frac{z - f(x)}{\sigma_\epsilon}\right]^2\right\}. \quad (23)$$

Based on the information given above, the prior estimate of the probability of failure for event $f(x, y) > 9$ and the prediction of system response z associated with a given reliability or confidence level can be calculated using FORM and inverse FORM methods. After obtaining the additional data $z = 8$, those estimates can be updated using the proposed analytical procedure. The updating process firstly involves the Laplace approximation for the posterior of Eq. (23). Then the iterative formula of Eq. (17) is employed to find the design point (x^*, y^*) and β, and the probability of failure P_F can be estimated using FORM according to $\Phi(-\beta)$.

To calculate the confidence bound (e.g., $[\text{lo}, \text{up}] = [0.025, 0.975]$ bound) of z, reliability indexes associated with the upper and lower limits are first calculated according to $\beta_{\text{lo}} = \Phi^{-1}(\text{lo})$ and $\beta_{\text{up}} = \Phi^{-1}(\text{up})$. The iterative inverse FORM formula of Eq. (20) solves the required design point for β_{lo} and β_{up}. Finally the confidence bound of z can be computed using these two design points. To compare the efficiency and the accuracy, MC and MCMC simulations serve as benchmark solutions to this example. Table 1 presents results for this example. The prior estimates for probability of failure (PoF) and interval prediction are calculated using FORM and inverse FORM methods, respectively. For this simple example, just a few function evaluations ensure obtaining converged results. A crude Monte Carlo simulation with 10^6 samples yields very close results. For the posterior estimate with Bayesian updating, the proposed analytical solutions using Laplace, FORM, and inverse FORM (results are labeled as iFORM in all the tables hereafter) methods are very close to the solution obtained using MCMC simulation with a chain length of 10^6. By comparing the number of function evaluations between the analytical and MC or MCMC solutions, it is observed that the proposed analytical method can reduce the computational cost by several orders of magnitude. It would be significantly advantageous to use the proposed analytical procedure for time constrained or online prognosis systems.

Table 1: Probability of Failure (PoF), confidence interval (CI) estimates, and the number of function evaluations (NFE) for $f(x, y) = x + y$. Both x and y are normally distributed with means of 2 and 5 and standard deviations of 0.5 and 1.5, respectively. The correlation coefficient between x and y is -0.5. The failure is defined as $f(x, y) > 9$.

	Method	PoF	95 CI	NFE
prior	FORM,iFORM	0.030181	4.2570~9.7430	21
	MC	0.030417	4.2493~9.7455	10^6
posterior	Laplace,FORM,iFORM	0.0045455	6.6536~8.9852	47
	MCMC	0.0046910	6.6506~8.9898	10^6

4.2 Reliability updating and response prognostics of a fatigue crack damaged system with experimental data

In this section, a practical fatigue crack damage problem is presented with experimental data. As a typical damage mode in many structures, the reliability of a system with possible fatigue cracks must be accurately quantified in order to avoid severe failure events. Because fatigue crack propagation is a time-dependent process, crack growth prognosis provides valuable information for system maintenance or unit replacement. Due to the stochastic nature of fatigue crack propagation, fatigue crack growth is not a smooth and stable process. Therefore additional information such as usage information from health monitoring systems and crack size measures from inspections can be used to update various quantities of interest. By performing continuous updating, uncertainties associated with system reliability and crack size prognosis can be reduced for decision-making. Because crack growth equations are usually in the forms of differential equations or finite element models, simulation-based methods are relatively more expensive in terms of computational cost. To demonstrate the updating procedure with the proposed method and validate its effectiveness and efficiency, experimental data are incorporated in this example. A portion of the experimental data is used to obtain the parameter distributions of the crack growth equation and one from the rest of the dataset is arbitrarily chosen to represent the "actual" target system. First we estimate PoF and crack growth prognosis with the prior parameter distributions. Then we choose a few points from the "acutal" target system to represent measurements from crack size inspections. These measures are used to perform Bayesian updating with the analytical methods proposed in previous sections. Both system reliability and crack growth prognosis are updated. Results are compared with simulation-based methods in terms of accuracy and efficiency.

(Virkler, Hillberry, & Goel, 1979) reported a large set of fatigue crack propagation data on aluminum alloy 2024-T3. The dataset consists of fatigue crack propagation trajectories recorded from 68 center-through crack specimens, each of which has the same geometry, loading, and material configurations. Each specimen has a width of $w = 154.2$mm and a thickness of $d = 2.54$mm. The initial crack size is $a_0 = 9.0$mm. A constant cyclic loading with a stress range of $\Delta\sigma = 48.28$MPa was applied. Without loss of generality, the classical Paris' equation (Paris & Erdogan, 1963) is chosen as the crack growth rate governing equation. Other crack growth equations can also be applied with the same procedure. Paris' equation describes the crack growth rate per one constant cyclic load as

$$\frac{da}{dN} = c(\Delta K)^m, \qquad (24)$$

where ΔK is the stress intensity range in one loading cycle.

For this particular crack and geometry configuration, $\Delta K = \sqrt{\pi a[\sec(\pi a/w)]}\Delta\sigma$. Terms c and m are uncertainty model parameters that are usually obtained via statistical regression analysis of experimental testing data. For convenience, $\ln c$ is usually used instead of c. Given a specific number of loading cycles, solving the ordinary differential equation in Eq. (24) gives the corresponding crack size.

The first fifteen crack growth trajectories from Virkler's dataset identifies these two parameters using Maximum Likelihood Estimation as a joint Gaussian distribution of $(\ln c, m)$ with a mean vector of $\mu_0 = [-26.7084, \ 2.9687]$ and a covariance matrix of $\Sigma_0 = \begin{bmatrix} 0.5435 & -0.0903 \\ -0.0903 & 0.0150 \end{bmatrix}$.

$$p_0(\ln c, m) = \frac{1}{2\pi\sqrt{|\Sigma_0|}} \times \\ \exp\left\{-\frac{1}{2}\left[(\ln c, m) - \mu_0\right]\Sigma_0^{-1}\left[(\ln c, m) - \mu_0\right]^T\right\} \qquad (25)$$

As we mentioned earlier in this section, another specimen from the rest of the dataset is arbitrarily chosen to represent the target system. The reliability and crack growth prognosis of this target system are of interest. The prior estimate of reliability and fatigue crack growth prognosis of the target system can then be estimated using this joint distribution and the model in Eq. (24). Let $\mathcal{M}(N; \ln c, m)$ denotes the model output (crack size) given a number of loading cycles N and parameters $\ln c$ and m. Three crack size measures a_i with corresponding numbers of loading cycles N_i at the early stage of the target system are chosen to represent the actual inspection data. They are $(a_1, N_1) = (10, 33062)$, $(a_2, N_2) = (11, 55101)$, and $(a_3, N_3) = (12, 75569)$. The standard deviation of Gaussian likelihood is also estimated as $\sigma_a = 0.304$mm. The failure event is defined as the crack size exceeding 40.0mm given the number of loading cycles as 220,000. With these additional measurement data, the posterior distribution of $(\ln c, m)$ (with r response measures) reads

$$p_n(\ln c, m) \propto p_0(\ln c, m) \times \\ \exp\left\{-\frac{1}{2}\sum_{i=1}^{r}\left[\frac{a_i - \mathcal{M}(N_i; \ln c, m)}{\sigma_a}\right]^2\right\} \qquad (26)$$

Following the proposed analytical procedure, we obtain updated results of reliability and crack size prognosis. Table 2 shows the prior and posterior (updated) results of PoF and 95% interval predictions of crack size at 220,000 loading cycles. We can observe from this table that the simulation method requires 200,000 function evaluations while the analytical method requires less than 200 function evaluations to produce similar results.

Figure 3 presents crack growth prognosis results obtained by the proposed analytical method. MCMC simulation results are displayed in the same figure for comparison. Several

Table 2: Prior and updated estimates of Probability of Failure (PoF), confidence interval (CI) for crack size, and the number of function evaluations (NFE) for fatigue crack problem. The failure is defined as the crack size exceeds $a_c = 40$mm at the number of loading cycles $N_c = 220,000$. 95% CI predictions are calculated at the number of loading cycles equal to N_c.

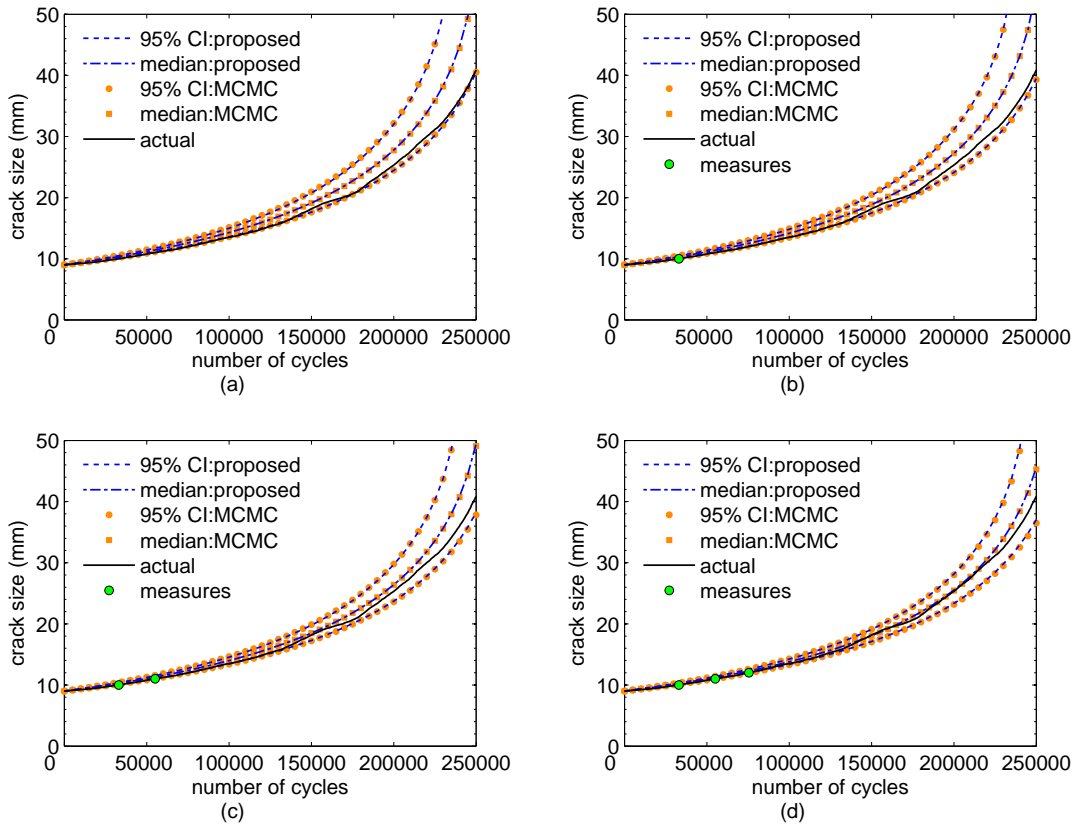| Measures	Method	PoF	95% CI	NFE
0(prior)	FORM,iFORM	0.0467	28.8290~41.3095	60
	MCMC	0.0498	28.9417~41.4685	2×10^5
1	Laplace,FORM,iFORM	0.0225	28.3694~39.8084	96
	MCMC	0.0186	28.3563~39.5466	2×10^5
2	Laplace,FORM,iFORM	0.0042	27.7926~37.4207	105
	MCMC	0.0039	27.6989~37.3537	2×10^5
3	Laplace,FORM,iFORM	0.0002	27.2484~34.9112	111
	MCMC	0.0001	27.0817~34.6913	2×10^5

Figure 3: Prior and posterior prognostics of fatigue crack growth trajectory using the proposed method (Laplace,iFORM) and the traditional simulation-based methods (MCMC). Median and 95% interval predictions are presented: (a) prior estimation; (b) updated with 1 measure; (c) updated with 2 measures; (d) updated with 3 measures.

aspects can be observed and interpreted: 1) The proposed analytical Laplace and (inverse) FORM method yields almost identical prognostic results to those obtained using traditional MCMC simulations, which can be confirmed by observing Figure 3(a-d). 2) In Figure 3(a), the prior median and interval prediction of the crack growth is far from the actual target system because of various uncertainty associated with the crack propagation process such as the material uncertainty, modeling uncertainty, as well as measurement uncertainty. These uncertainties are finally encoded into the model parameter $(\ln c, m)$ in form of distributions through statistical regression.

These uncertainties cause the prior estimation deviates from the actual target system. 3) Inspection data, or crack size measurement in this example, is critical to improve the accuracy for time-dependent nonlinear system prognostics. With inspection data, uncertainties can be greatly reduced. As shown in Figure 3(b-d), both the median and interval predictions for crack growth trajectories become closer to the actual trajectories as more measurements are integrated into the Bayesian updating process.

4.3 A cantilever beam example

A beam example is used to examine the proposed method through finite element analysis (FEA). Data from FEA provide representative sensor output. By analyzing the sensor output data, frequency information of the beam is extracted to update the finite element model and also the reliability level. For the sake of illustration and simplification, we use a simple cantilever beam. More complex full-scale structural finite element model analysis follows the same procedure as presented here.

A cantilever aluminum beam is divided into ten elements using finite element modeling, as shown in Figure (4). The beam is 1m long, 0.1m wide and 0.01m thick. The design cross section area is $A = 0.001\text{m}^2$. Assume the cross section area of the first segment of the beam (attached to the wall) is modeled by $A_1 = \alpha A$ due to manufacturing uncertainty, where term α is a Gaussian variable with a mean of 1 and a standard deviation of 0.5. Because of usage (aging) and material degradation, α may vary along time. Other segments have deterministic cross section dimensions that are equal to the design value of A. The material has a Young's modulus of $E = 6.96 \times 10^{10}\text{Pa}$ and a density of $2.73 \times 10^3\text{kg/m}^3$.
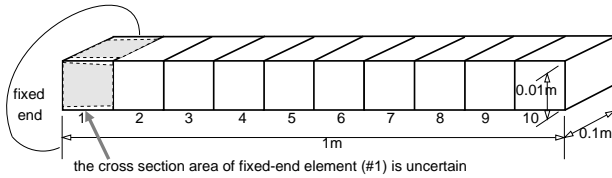

Figure 4: The cantilever beam finite element model. The cross section area of the first element (attached to the wall) is uncertain due to manufacture and usage and is modeled by $A_1 = \alpha A$, where $A = 0.001\text{m}^2$ is design cross section area and $\alpha \sim \text{Norm}(1, 0.02^2)$.

The failure event is defined as the first natural frequency is less than 8Hz due to the degradation of the stiffness of the beam. The sensor data are synthesized by setting $\alpha = 0.95$ and solving the dynamical equation of the beam under a free vibration. After adding 5 percent of Gaussian white noise, the first four mode frequency data are extracted from the sensor data using Fast Fourier Transformation (FFT). They are $(f_1, f_2, f_3, f_4) = (8.03, 50.5, 142, 280)Hz$.

Based on the above information, the Bayesian posterior for uncertain variable α given the frequency information extracted from the sensor data is

$$p(\alpha) \propto \exp\left\{-\frac{1}{2}\frac{(\alpha-1)^2}{0.5^2}\right\}$$
$$\times \exp\left\{-\sum_{j=1}^{F}\sum_{i=1}^{N}\left(\{\omega\}_i\left[\left(-(2\pi f_i)^2\mathbf{M}(\alpha) + \mathbf{K}(\alpha)\right)\{\phi_i\}\right]_j\right)^2\right\}, \quad (27)$$

where N is the number of measured mode and F is the number of measured mode shape coordinates. Term $\{\omega\}_i$ is the ith weighting factor for ith frequency component in the likelihood function. For the purpose of illustration, $\{\omega\}$ is configured such that each frequency component has a coefficient of variation of 0.1. Terms $\mathbf{M}(\alpha)$ and $\mathbf{K}(\alpha)$ are the mass and stiffness matrices, respectively. Because α is a variable, actual values for $\mathbf{M}(\alpha)$ and $\mathbf{K}(\alpha)$ depends on each realization of α. Term $\{\phi\}_i$ is the ith mode shape. For the current data, $N = 4$ and $F = 20$.

Using the proposed method we obtain results shown in Table (3). Simulation-based results are also listed in this table for comparison.

Table 3: Prior and posterior estimates of probability of failure (PoF) in the beam example. Frequency data (first four natural frequency extracted from synthesized noisy data via FFT) are used to perform Bayesian updating. Statistics of α (mean,standard deviation(SD)) and computational cost in term of number of function evaluations (NFE) are shown.

	Method	PoF	NFE	α(mean,SD)
prior:	FORM	0.004435	11	1.0, 0.02
	MC	0.004428	10^6	1.0, 0.02
posterior:	Laplace,FORM	0.0182	57	0.9757, 0.0134
	MCMC	0.0164	10^6	0.9760, 0.0135

Results of the proposed method are similar to those obtained using traditional simulation-based methods. However, the computational cost is much smaller. Finite element models in practical problems are usually more sophisticated than this beam example, and simulation-based methods are not feasible for such computationally extensive problems. The proposed method provides an alternative to solving such problems and it yields accurate results under the condition that uncertain variables are approximately Gaussian-like.

In this section, three examples are presented to demonstrate and validate the proposed analytical method. Some important aspects of the proposed method are closely revealed, including the computational benefits in terms of efficiency and accuracy. Appropriate conditions to assure these benefits are also analyzed.

5. CONCLUSIONS

In this paper, an efficient analytical Bayesian method for reliability and system response updating is developed. The method is capable of incorporating additional information such as inspection data to reduce uncertainties and improve the estimation accuracy. One major difference between the proposed work and the traditional approach is that the proposed method performs all the calculations including Bayesian updating without using MC or MCMC simulations. A numerical example, a practical fatigue crack propagation problem with experimental data, and a finite element beam problem with FEA data are presented to demonstrate the proposed method. Comparisons are made with traditional simulation-based methods to investigate the accuracy and efficiency. Based on the current study, several conclusions are drawn.

1. The proposed method provides an efficient analytical computational procedure for computing and updating system reliability responses. No MC or MCMC simulation is required therefore it provides an feasible and practical solution to time constrained or online prognostics. The method is also beneficial for structural health monitoring problems where Bayesian updating and system response predictions are frequently performed upon the arrival of sensor data.

2. The proposed method is capable of incorporating additional information such as the inspection data and usage data from health monitoring system by way of Bayesian updating. This property is beneficial for highly stochastic time-dependent nonlinear system where prior estimates for reliability and system response may become unreliable along with

system developing. By continuous Bayesian updating, estimation uncertainties can be reduced.

3. The proposed method yields almost identical results to those produced by traditional simulation-based methods given that uncertain variables are approximately Gaussian distributed. This is true for most of the engineering problems where the uncertain parameters are normal or log-normal variables (which can be transformed and truncated into normal variables). When these conditions are not assured, the results need careful interpretations. The efficiency and accuracy of the proposed method is demonstrated and verified using three examples. The proposed method provides an alternative for time-constrained prognostics problems. If the problem involves too many random variables, traditional simulation-based method may be more appropriate. Systematical comparisons of the method with other approaches such as variational method will be conducted in the future.

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