Evaluating Failure Time Probabilities for Compound Degradation with Linear Path and Mixed Jumps

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ABSTRACT

Many devices may experience nature degradation and mixed jumps simultaneously whose types can be divided into positive jumps and negative jumps, while these complicated performance rules also bring difficulties in lifetime analysis within the concept of the first hitting time. To address this issue, this paper first proposes a compound degradation process, which is characterized by linear path and mixed jumps. Then, by adopting the idea that transforms the positive jumps into the threshold, an approximate lifetime solution is derived. Given the realistic application of furnace wall, numerical verification shows that the proposed method can maintain consistency with Monte Carlo simulation, while conspicuous errors exist for existing methods, demonstrating that the proposed method can be regarded as theoretical support for the future studies.

Keywords: Degradation; Mixed jumps; First hitting time; Analytical lifetime distribution.

1. INTRODUCTION

The past decades have seen the rapid development of degradation modeling and lifetime analysis since the devices universally experience inevitable deterioration, which results in failure, even catastrophes (Li et al., 2017; Wang et al., 2023; Zhang et al., 2023). Attributed to the complex internal and external factors, the performance rules have gradually evolved from continuous degradation to discontinuous one, which contains random arrival jumps, e.g., positive jump and negative jump, illustrating unstable characteristics. For the purpose of prescribing replace polices and predicting remaining useful life, developing accurate degradation modeling and lifetime assessment for such type of degradation processes are apparent.

Attributed to the fact that the positive jump remains the monotonic traits of natural degradation, its lifetime solution has been extensively studied (Che et al., 2018; Fan et al., 2017; Kharoufeh et al., 2006). The experiment on MEMS micro-engines, initiated by Sandia, found that external shock causes abrupt positive jump (Tanner et al., 2000), which is also known as the primary contributing factor for other devices. According to this phenomenon, Peng et al. (2011) considered that the arrival positive jump obeys a Poisson process, and the total degradation presented a linear path with positive jumps. To a further step, Wei et al. (2019) found that the magnitude of positive jump is proportional to external shock, and divided the shock into harmless, harmful, and fatal ones, arising non-happening jump, positive jump and direct failure events respectively.

Along with the deep-going of the research, it is found that the negative jump appears under the influence of self-healing mechanism or maintenance activities (Charri & Vinassa, 2014; Wang et al., 2023), and the total degradation performs non-monotonicity and randomness simultaneously. Obviously, evaluating its failure time probability is nontrivial but hard to derive a closed-form solution, and numerous studies turn to use intelligence algorithm to evaluate the lifetime (Sun et al., 2023; Xu et al., 2024). Obviously, adopting such method will be overwhelmed by heavily computing burden, rendering short-term prediction of useful life unattainable. Inspired by the lifetime analysis method proposed for positive jump, Wang et al. (2018) transformed the negative jumps into the failure threshold, and derived an approximate lifetime solution. However, such strong approximate assumption arises significant evaluation errors in some certain cases. To obtain a high-efficient method, Cao et al. (2025) innovatively proposed the concept of invalid epoch, and derived an analytical lifetime solution in the form of Laplace-Stieltjes transform (LST).

The mixed jump is a more complex situation, which is a combination of positive and negative jumps. Hindered by the above challenges, most studies still rely on simulation methods to calculate the lifetime, with effective analytical lifetime analysis techniques being quite rare. However, such phenomena are far from trivial and are increasingly encountered in various scenarios. The most typical situation is the deterioration of rotating mechanical component with multiple operational profiles. Li et al. (2019) and Kong et al. (2021) found that when dynamic environment is involved, the vibration signals of both bearings and CNC milling machine are composed of nature degradation and mixed jumps. The similar phenomenon is also found in the devices with complex failure mechanisms. For instance, for the furnace wall, due to the complex chemical reactions between molten iron and furnace wall, the measured temperature signal consists of nature degradation and mixed jumps (Zhang et al., 2017).

As reviewed above, one can conclude that the degradation process with abrupt jumps has become a focus in recent years, while there still exist some shortcomings. The prominent problem is that although the lifetime analysis method has been readily solved for linear path with negative jumps (Cao et al., 2025), there remains a gap in the analysis for mixed jump scenarios. The possibility to extend the current theory for conducting lifetime analysis for mixed jumps presents an intriguing topic. To address this issue, by converting the positive jump into the threshold, this paper proposes a closed-form approximate lifetime solution for linear path with mixed jumps.

The remainder of the paper is organized as follows. Section 2 delineates the scope of the model and establishes the compound degradation process involving linear path with mixed jumps. Section 3 derives an analytical lifetime solution under the concept of the first hitting time (FHT). Then, Section 4 conducts the numerical verification via a realistic application. Finally, Section 5 summarizes the main conclusions.

2. MATHEMATICAL MODEL

This section focuses on the establishment of the compound degradation process with linear path and mixed jumps. As plotted in Fig. 1, the product undergoes continuous degradation process and five discrete jumps. Due to the existence of mixed jumps, the overall degradation exhibits non-monotonicity and fluctuations, and once the total degradation first hits the preset failure threshold x, then the failure happens at T_x, which is also defined as the first hitting time (FHT). To delineate the model with clarity, the following assumptions are introduced.

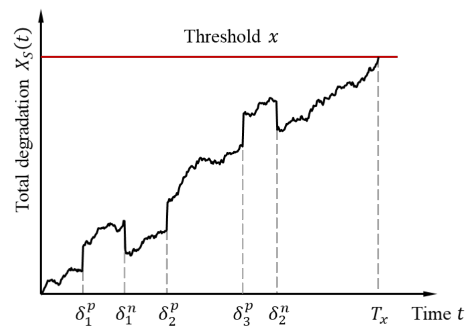

Figure 1. Schematic diagram of linear path with mixed jumps.

(1) The continuous process is described by a linear path

$$X(t) = rt + \varepsilon \tag{1}$$

where r is the degradation rate, and the random error is assumed to obey the normal distribution, denoted as $\varepsilon \sim N(\mu_\varepsilon, \sigma_\varepsilon^2)$.

(2) The arrival times of positive jump are decided by a Poisson process with intensity λ^p, corresponding to a time sequence $\{\delta_i^p\}_{i=1}^{\infty}$, and its arrival number until t is recorded as $N^p(t)$. Besides, to capture a variety of situations, it is assumed that the magnitude of positive jump follows non-restrictive CDF $F_Y^p(y)$ with domain $[0, +\infty)$, which constitutes the sequence of $\{Y_i^p\}_{i=1}^{\infty}$. Under the above assumptions, the cumulative magnitude of positive jump until t is

$$S_p(t) = \sum_{i=1}^{N^p(t)} Y_i^p \tag{2}$$

(3) The procedure concerning positive jumps operates independently of those related to negative jumps. Similar to those of positive jump events, the sequence $\{\delta_j^n\}_{j=1}^{\infty}$ of negative jump arrival times follow a Poisson process with intensity λ^n, and their magnitudes consist the nonnegative sequence $\{Y_j^n\}_{j=1}^{\infty}$ based on non-restrictive CDF $F_Y^n(y)$. Thus, the cumulative magnitude of negative jump up to t is

$$S_n(t) = \sum_{j=1}^{N^n(t)} Y_j^n \tag{3}$$

where $N^n(t)$ represents the arrival number of negative jumps until t.

(4) The compound degradation process is consisted of three parts including continuous degradation, positive jumps and negative jumps. Thus, we have

$$X_S(t) = X(t) + S_p(t) - S_n(t) \tag{4}$$

Due to the randomly occurring negative jumps, the overall degradation exhibits non-monotonicity. To guarantee the reliability of the product, it is usually announced that the product fails when its total degradation first hits the failure threshold, and the failure time is known as the FHT. Given the failure threshold x, the formal definition is:

$$T_x = inf\{t \vee X_S(t) \geq x\} \tag{5}$$

The corresponding FHT distribution is

$$F(x,t) = Pr\{T_x \leq t\} \tag{6}$$

There is no deny that the FHT distribution plays a key part in remaining useful life prediction and maintenance decision-making, and we will place our primary focus on the FHT analysis for the proposed model.

3. LIFETIME ASSESSMENT

Before deriving the analytical FHT solution, some important mathematical prerequisites are introduced. As a well-known mathematical technique, the Laplace-Stieltjes transform (LST) for the FHT function with respect to variable t is defined as

$$\widetilde{F}(x,s) = \int_0^\infty e^{-st} F(x, dt) \tag{7}$$

The current study has investigated the FHT distribution of linear path with negative jumps, and has derived a novel analytical lifetime solution in the form of LST, which is regarded as a key basis of our work. To be specific, it yields:

Lemma (Cao et al., 2025). For the compound degradation with linear path and negative jumps, if the PDF of cumulative magnitude $f_{S_n(\upsilon)}(z \vee N^n(\upsilon) = j)$ exists for $\forall\, j \in N$, and $\forall\, \upsilon \in [0, +\infty)$, then

$$\widetilde{F}(x, s \vee \lambda^p = 0) \cong \exp\left\{\left[\frac{\lambda^n}{r}\psi(s) - \frac{s}{r} - \frac{\lambda^n}{r}\right]x\right\} \tag{8}$$

where $\psi(s) = \sum_{j=1}^{\infty} \int_0^\infty P_j \cdot e^{-s \cdot z/r} \cdot f_{S_n(\upsilon)}(z \vee N^n(\upsilon) = j)\, dz$, $P_j = \frac{e^{-\lambda^n z/r}(\lambda^n z/r)^{j-1}}{j!}$, and $f_{S_n(\upsilon)}(z \vee N^n(\upsilon) = j)$ is the PDF of the cumulative magnitude under the condition of $N^n(\upsilon) = j$.

The above lemma holds true when $x \geq 0$, and obviously, $\widetilde{F}(x,s) = 1$ when $x < 0$. To ensure a unified expression, we define

$$\widetilde{F}_1(x,s) = \begin{cases} 1 & x < 0 \\ \widetilde{F}(x, s \vee \lambda^p = 0) & x \geq 0 \end{cases} \tag{9}$$

Based on the aforementioned conclusions, the idea of threshold transformation is employed as an approximate approach to address the FHT solution of the proposed model, and the following theorem is proposed.

Theorem. Let $L^{-1}\{\cdot\}$ denotes the inverse Laplace transform. If the PDFs of $f_{S_p(\upsilon)}(z \vee N^p(\upsilon) = i)$ and $f_{S_n(\upsilon)}(z \vee N^n(\upsilon) = j)$ exist for $\forall\, i, j \in N$, and $\forall\, \upsilon \in [0, +\infty)$, then the FHT distribution can be approximated as

$$\begin{aligned} &F(x,t) \\ &\cong \sum_{i=0}^{\infty} L^{-1}\left\{ s^{-1} \cdot \int_0^\infty \widetilde{F}_1(x-z, s) \cdot f_{S_p(t)}(z \vee N^p(t) = i)\, dz \right\} \\ &\cdot Pr\{N^p(t) = i\} \end{aligned} \tag{10}$$

where $Pr\{N^p(t) = i\} = \frac{e^{-\lambda^p t}(\lambda^p t)^i}{i!}$.

Proof. Given the condition of $N^p(t) = i$, the arrival positive jumps can be transformed into the threshold, and the lifetime distribution is rewritten as

$$\begin{aligned} &F(x, t \vee N^p(t) = i) \\ &\text{¿}\, Pr\{X(t) - S_n(t) \leq x - S_p(t) \vee N^p(t) = i\} \end{aligned} \tag{11}$$

Note that the right side decrease monotonically, and for most of actual degradation processes, the rate of decrease on the left side is greater than that on the right side on average, which means that it can be approximately considered that there is only one intersection point between the two. Then Eq. (11) is approximated as

$$F(x,t \vee N^p(t)=i) \cong \int_0^\infty F_1(x-z,t) \cdot f_{S_p(t)}(z \vee N^p(t)=i)\, dz \tag{12}$$

where $F_1(x,t)=L^{-1}\{s^{-1} \cdot \widetilde{F}_1(x,s)\}$.

Based on the Poisson process, we have

$$Pr\{N^p(t)=i\}=\frac{e^{-\lambda^p t}(\lambda^p t)^i}{i!} \tag{13}$$

Combining the results of Eq. (12) and Eq. (13), the following approximation holds

$$F(x,t) \cong \sum_{i=0}^\infty L^{-1}\left\{s^{-1} \cdot \int_0^\infty \widetilde{F}_1(x-z,s) \cdot f_{S_p(t)}(z \vee N^p(t)=i)\, dz \cdot Pr\{N^p(t)=i\}\right. \tag{14}$$

and the proof is complete.

Since the above findings holds without constraints on the CDFs of $F_Y^p(y)$ and $F_Y^n(y)$, it can be regarded as a general solution for practical applications. More importantly, when both positive jumps and negative jumps follow the normal distributions, i.e., $Y_i^p \;\; N(\mu_p, \sigma_p^2)$ and $Y_i^n \;\; N(\mu_n, \sigma_n^2)$, the expression can be further simplified; for detail derivation, readers can combine the proposition proposed by Cao et al. (2025), and we shall not go into much detail.

4. APPLICATION CASE

In this section, the quality and the applicability of the main results are demonstrated via numerical experiment. Specifically, we consider the erosion of furnace wall, and the existing methods are applied as comparisons.

4.1. Background

The furnace wall is a large-scale complex system that inevitable experiences degradation due to the erosive effects of molten iron. However, within the background of continuous monitoring, determining the loss in wall thickness poses significant challenges. As an indirect alternative measurement approach, temperature sensors are installed at a proper depth of the furnace wall to reflect its operational status. Obviously, the conditional monitored temperature will continuously rise, and more importantly, furnace wall may undergo chemical reactions with the molten iron wall, resulting in the change of thermal conductivity of the furnace wall. As a result, the measured temperature exhibits random fluctuations, and thus the total degradation procedure is composed of continuous degradation and mixed jumps. To guarantee the reliability and the safety, it is regulated that when the monitored temperature first exceeds a specified threshold, the furnace wall must be shut down or replaced. Based on the experts' experience and previous studies (Zhang et al., 2017), the specific model parameters for furnace wall are given in Table 1.

Table 1. Model parameters for furnace wall.

Parameter	Value	Description
x	$500\,℃$	The failure threshold.
r	$1\,℃/day$	The degradation rate.
λ^n	$2\times10^{-2}/day$	The Poisson intensity of negative jumps.
λ^p	$2\times10^{-2}/day$	The Poisson intensity of positive jumps.
Y_i^n	$Y_i^p \;\; N(30,10^2)$	The CDF of magnitudes of negative jumps.
Y_i^p	$Y_i^n \;\; N(50,10^2)$	The CDF of magnitudes of positive jumps.

4.2. Main results

According to the parameters outlined in Table 1, the FHT distribution for the proposed model can be calculated via inverse Laplace algorithm (Abate & Whitt, 1995). As comparisons, the current models are introduced, which are abbreviated as follows:

M_0: The model proposed in our work that incorporates the linear path and mixed jumps simultaneously. The FHT distribution is calculated via Theorem 1.

M_1: Only the linear path and negative jumps are considered, and the FHT distribution can be obtained according to Lemma 1 (Cao et al., 2025).

M_2: Its total degradation is composed of linear path and positive jumps, and the FHT distribution is given by Peng et al. (2011).

Meanwhile, we shall adopt Monte Carlo simulation as a benchmark to verify the theoretical correctness. The simulation generates the pseudo lifetime in accordance with the degradation model proposed in Section 2, where the sample size and discretized time interval of simulation are preset as $M=50000$, and $\triangle t=0.1$.

As shown in Fig. 2, the FHT results for simulation, M_0, M_1 and M_2 are plotted. One can see that the proposed method is consistent with the simulation during the whole computing period. In contrast, inevitable errors appear for both comparison methods M_1 and M_2. The reason arises from the incomplete consideration of the types of jumps. To evaluate the performance of the proposed model quantitatively, two important lifetime indexes including the maximum absolute CDF error and the relative error of MTTF are calculated, where $MTTF=\int_0^\infty t\cdot f(x,t)\,dt$, and $f(x,t)$ is the PDF of FHT. Results show that, for our method M_0, the maximum absolute CDF error is 3.53% and the relative error of MTTF is 2.41%. Consequently, comprehensive considerations are required when both negative jumps and positive jumps exist.

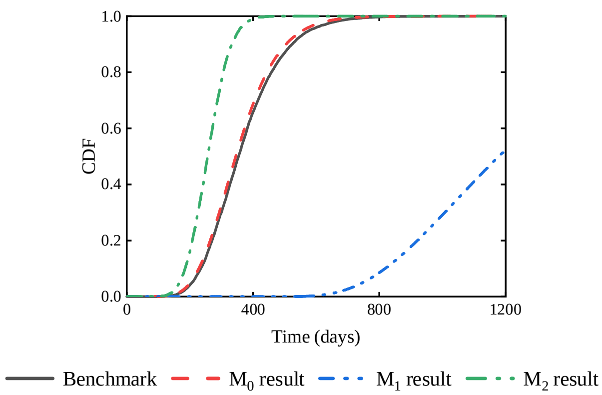

Figure 2. Lifetime results of simulation and analytical methods.

5. CONCLUSION

Owing to the complexity of operational factors growing, the degradation modeling and lifetime analysis for discontinuous degradation have been increasingly focused on, especially when positive jumps and negative jumps (mixed jumps) are presented. In this paper, we develop a novel compound degradation process involving linear path and mixed jumps. Then, by transforming positive jumps into the threshold, an approximate FHT distribution is derived. To verify the rationality and the practicality of the method, a real application of furnace wall is demonstrated. Results show that compared with current methods, the proposed method performs better. Although numerical studies demonstrate the proposed method, the solution is still preliminary, and under the condition of significant magnitude or frequent jump, the approximation may involve inevitable errors. Thus, in the future studies, how to improve the accuracy of the FHT solution is an important issue.

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REFERENCES

Abate, J., & Whitt, W. (1995). Numerical inversion of Laplace transforms of probability distributions. *OSRA Journal on Computing.* 7 (1) 36-43. doi:10.1287/ijoc.7.1.36.

Cao, S. H., Wang, Z. H., Wu, Q., Ouyang, Xiangmin., Si, X. S., & Liu, C. R. (2025). Failure time analysis for compound degradation procedures involving linear path and negative jumps. *Reliability Engineering & System Safety.* 253. doi:10.1016/j.ress.2024.110566.

Charri, R., Briat, O., & Vinassa, J. M. (2014). Capacitance recovery analysis and modelling of supercapacitors during cycling ageing tests. *Energy Conversion and Management.* 82, 37-45. doi:10.1016/j.enconman.2014.02.051.

Che, H. Y., Zeng, S. K., Guo, J. B., & Wang, Y. (2018). Reliability modeling for dependent competing failure processes with mutually dependent degradation process and shock process. *Reliability Engineering & System Safety.* 180, 168-178. doi:10.1016/j.ress.2018.07.018.

Fan, M. F., Zeng, Z. Z., Zio., E., & Kang, R. (2017). Modeling dependent competing failure processes with degradation-shock dependence. *Reliability Engineering & System Safety.* 165, 422-430. doi:10.1016/j.ress.2017.05.004.

Kharoufeh, J. P., Finkelstein, D. E., & Mixon, D. G. (2006). Availability of periodically inspected systems with markovian wear and shocks. *Journal of Applied Probability.* 43 (2), 303-317. doi:10.1239/jap/1152413724.

Kong, X. F., Yang, J., & Li, L. (2021). Remaining useful life prediction for degrading systems with random shocks considering measurement uncertainty. *Journal of Manufacturing Systems.* 61, 782-798. doi:10.1016/j.jmsy.2021.05.019.

Li, J. X., Wang, Z. H., Zhang, Y. B., Fu, H. M., Liu, C. R., & Krishnaswamy, S. (2017). Degradation data analysis based on a generalized Wiener process subject to measurement error. *Mechanical Systems and Signal Processing*. 94, 57-72. doi:10.1016/j.ymssp.2017.02.031.

Li, N. P., Gebraeel, N., Lei, Y. G., Bian, L. K., & Si, X. S. (2019). Remaining useful life prediction of machinery under time-varying operating conditions based on a two-factor state-space model. *Reliability Engineering & System Safety*. 186, 88-100. doi:10.1016/j.ress.2019.02.017.

Peng, H., Feng, Q. M., & Coit, D. W. (2011). Reliability and maintenance modeling for systems subject to multiple dependent competing failure processes. *IIE Transactions*. 43 (1), 12-22. doi:10.1080/0740817X.2010.491502.

Sun, C., Qu, A., Zhang, J., Shi, Q. Y., & Jia, Z. H. (2023). Remaining useful life prediction for lithium-ion batteries based on improved variational mode decomposition and machine learning algorithm. *Energies*, 16 (1). doi:10.3390/en16010313.

Tanner, D. M., Walraven, J. A., Helgesen, K., Irwin, L. W., Brown, F., Smith, N. F., & Masters, N. (2000). MEMS reliability in shock environments. *2000 IEEE International Reliability Physics Symposium Proceedings* (129-138), April 10-13, San Jose, CA, USA. doi:10.1109/RELPHY.2000.843903.

Wang, H. W., Xiang, J. W., Zhao, X. F., & Li, Y. L. (2019). A kinematic precision reliability evaluation method for rotor-bearing systems considering multi-source wear degradations and random errors. *The International Journal of Advanced Manufacturing Technology*. 124 (11-12), 4159-4173. doi:10.1007/s00170-022-09383-x.

Wang, W. K., & Chen, X. (2023). Piecewise deterministic Markov process for condition-based imperfect maintenance models. *Reliability Engineering & System Safety*. 236. doi:10.1016/j.ress.2023.109271.

Wang, Z. Q., Hu, C. H., Si, X. S., & Zio, E. (2018). Remaining useful life prediction of degrading systems subjected to imperfect maintenance: Application to draught fans. *Mechanical Systems and Signal Processing*. 100, 802-813. doi:10.1016/j.ymssp.2017.08.016.

Wei, G. Z., Zhao, X. J., He, S. G., & He, Z. (2019). Reliability modeling with condition-based maintenance for binary-state deteriorating systems considering zoned shock effects. *Computers & Industrial Engineering.* 130, 282-297. doi:10.1016/j.cie.2019.02.034.

Xu, Z. C., Xie, N. M., & Li, K. L. (2024). Remaining useful life prediction for lithium-ion batteries with an improved grey particle filter model. *Journal of Energy Storage*, 78. doi:10.1016/j.est.2023.110081.

Zhang, J. X., Hu, C. H., He, X., Si, X. S., Liu, Y., & Zhou, D. Y. (2017). Lifetime prognostics for deteriorating systems with time-varying random jumps. *Reliability Engineering & System Safety*. 167, 338-350. doi:10.1016/j.ress.2017.05.047.

Zhang, Z. H, Zhao, C. W, Zhao, Z. P., Wang, F. M., & Zhao, B. (2023). Structural fatigue reliability evaluation based on probability analysis of the number of zero-crossings of stochastic response process. *Engineering Failure Analysis.* 143. doi:10.1016/j.engfailanal.2022.106923.

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