Quasi-Instantaneous Battery End-of-Discharge Time Prognosis with Non-Stationary Autoregressive Exogenous Inputs

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Abstract

Reliable automation of engineering systems consider the implementation of condition monitoring routines to make decisions that can be carried out either manually, or through control loops. In recent years, great efforts have been made to incorporate prognostics into decision making, primarily through heuristic or data-driven approaches. However, being effective and efficient in this task is quite difficult. On one hand, it takes time to compute predictions, which makes prognostics-informed control unfeasible in real-time. On the other hand, until very recently, prognostics lacked a solid theoretical foundation to provide formalism and mathematical guarantees on prediction convergence. The concept of near-instantaneous prognosis was recently introduced, enabling prognostic analysis in nonlinear dynamical systems with maximal performance and stochastic convergence guarantees, under the assumption that future exogenous inputs follow strictly stationary stochastic processes. In this article, near-instantaneous prognosis is revisited to predict the End-of-Discharge time of batteries in which the discharge current, interpreted as exogenous input, is modeled as an autoregressive stochastic process. Furthermore, the strict stationarity condition is relaxed giving rise to a new variant presented as quasi-instantaneous prognosis, seeking to generalize this approach by removing this restrictive hypothesis in order to broaden its range of applications, specially in electrical engineering, such as in solar and wind resource prediction, or energy demand forecasting, to name some examples.

1. Introduction

Undoubtedly, one of the biggest challenges in automation and decision-making regarding an engineering system is being able to monitor its condition and predict the future consequences of actions. This translates in risk that must be characterized and quantified effectively and efficiently to avoid the occurrence of undesired events (Vachtsevanos & Zahiri, 2022). Predictive maintenance in Industry 4.0 and 5.0 emerges as a possible application (Ahmed Murtaza et al., 2024). In this regard, there have been great advances in the development of prognostic algorithms, largely due to the advances made in the field of artificial intelligence and machine learning (Biggio & Kastanis, 2020). These advances seem promising in terms of metrics associated with prediction (Fink et al., 2020), however, there are still fundamental discussions that remain open in this regard. This context can be explained by the lack of a formal theoretical framework for prognosis theory (Acuña-Ureta, Orchard, & Wheeler, 2021). Both practical and theoretical limitations remain.

One of the areas where forecasting becomes critical is the energy sector, particularly in demand prediction (Ghazal et al., 2022). Energy generation must respond dynamically to demand; however, effective generation management requires accurate anticipation to ensure timely supply and prevent shortages. Beyond increasing environmental awareness, the energy landscape has evolved with the growing integration of clean and renewable sources, such as solar and wind power. These resources, however, exhibit inherent stochasticity, making their output highly irregular and challenging to predict (W. Wang, Yuan, Sun, & Wennersten, 2022), so the energy produced is stored in batteries to be accessed whenever needed. In other words, an energy management system combines different generation sources so that, in the event of a generation deficit or excess, batteries can store or supply energy (Rana et al., 2023).

Optimal and efficient energy management depends greatly on the context and the merit functions involved, and demand forecasting is key in this regard. In a national electric power system (Alabi et al., 2022). For example, energy demand forecasts can inform the timely activation of auxiliary generation units, such as combined-cycle plants, which require a lead time to become operational (Van den Bergh et al., 2017),

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or to predict market prices and decide whether to store or supply energy conveniently to maximize profits (Shamsi & Cuffe, 2022). In contrast, when microgrids are deployed in isolated communities, typically at a smaller scale than large utility systems, forecasting is often aimed at rationing scarce energy resources to ensure reliability (Wazirali, Yaghoubi, Abujazar, Ahmad, & Vakili, 2023). Finally, there are smaller-scale examples, such as the energy of an electric vehicle (Xiong, Cao, Yu, He, & Sun, 2017) or an autonomous robot (Schacht-Rodríguez, Ponsart, García-Beltrán, & Astorga-Zaragoza, 2018), where energy autonomy is critical for operation. In such cases, predicting discharge time is essential for adapting the operating profile, such as adjusting navigation aggressiveness, to manage recharging and determine autonomy limit (García Bustos et al., 2025). The contributions of this article are framed in this context, where predicting battery discharge time becomes essential.

1.1. End-of-Discharge time prognosis

Advances in battery monitoring methods (Dong, Shen, Sun, Zhang, & Wei, 2025) are critical to the reliability of autonomous systems, especially for considering effective operational time in decision-making (Y. Wang, Tian, et al., 2020). Typically, battery-powered devices are programmed by firmware to shut down if voltage levels fall below a certain cut-off value, as exceeding this limit could damage the battery (Maden & Arabacı, 2024). Since the State-of-Charge (SoC) cannot be measured directly (Dang, Yang, Liu, & Chen, 2024), the battery terminal voltage is considered as an indicator. Given a current SoC, the predicted End-of-Discharge (EoD) time is defined as the future time at which the cut-off voltage is reached (Chen, Sun, Dong, Wei, & Wu, 2019).

In (Saxena et al., 2012), a geometrical parametric structure is assumed for a voltage curve, whose parameters are estimated following a data-driven approach. Under a Bayesian filtering approach, particle-filtering-based prognostics (Orchard & Vachtsevanos, 2009) are applied in (Pola et al., 2015) to compute EoD time probability distribution assuming that discharge currents follow the statistics of a two-state Markov Chain. In terms of accuracy, particle-filtering-based prognostic algorithms in (Walker, Rayman, & White, 2015) are shown to outperform other approaches based on the Unscented Kalman Filter (UKF) and non-linear least squares. The same stochastic characterization of exogenous inputs is employed in (Acuña-Ureta & Orchard, 2022), but the prognostic algorithm is changed by introducing the concept of near-instantaneous prognosis. An adaptive EoD time prognostics method is proposed in (Sbarufatti, Corbetta, Giglio, & Cadini, 2017), which uses radial basis function neural networks whose parameters are estimated in real-time. The authors in (Y. Wang, Gao, Li, & Chen, 2020) developed fractional-order models and introduced a Bayesian Monte Carlo estimator for EoD time estimation, using future load trajectory information derived from a Markov probability model for such a purpose. Using accurate battery models and the UKF for state estimation, another way of characterizing future battery discharge currents based on the discrete wavelet transform technique is employed in (Dong, Wei, Chen, Sun, & Yu, 2017), where input currents' past dynamics are assumed as known and forgetting factors incorporated recursively also allow constant current predictions. In (X. Zhang, Wang, Liu, & Chen, 2017), the Recursive Least Squares (RLS) and UKF algorithms are applied to perform parameter estimation of battery models and EoD time prognostics, respectively. A reduced-order electrochemical model in combination with the Lambert function are used to predict EoD time in (Quiñones, Milocco, & Real, 2018). Accounting for hysteresis phenomena, a new framework based on the Unscented Particle Filter (UPF) is proposed in (Y. Wang & Chen, 2020) for SoC estimation and EoD time estimation. In another study (Jinsong, Shuang, Diyin, & Hao, 2017), a Dirichlet process mixture model was developed to identify voltage trajectories, which can then be used for EoD time prognostics. A probabilistic approach based on Monte Carlo simulations and the Akaike information criterion is proposed in (Y. Zhang, Xiong, He, & Pecht, 2019) for uncertainty characterization and quantification in SoC estimation and future load predictions.

1.2. Autoregressive models and future discharge prediction

Naturally, the prediction of the EoD time depends on the manner in which the battery is discharged. That is, how the discharge current evolves over time. How can this prediction be made if this evolution is unknown? It therefore makes sense to extrapolate historical statistical behavior into the future. This historical pattern of battery use is known as *"discharge profile"*, which can be characterized through a stochastic process for the discharge current in various ways. As mentioned above, this characterization can be done through Markov Chains (Pola et al., 2015; Acuña-Ureta & Orchard, 2022). However, this article explores autoregressive models to perform the characterization, since they are widely known and applied in this type of context (M. Amini, Karabasoglu, Ilić, Boroojeni, & Iyengar, 2015; M. H. Amini, Kargarian, & Karabasoglu, 2016; Khalid, Sundararajan, & Sarwat, 2019; Kim, Son, & Kim, 2019; Khalid & Sarwat, 2021; Kim & Kim, 2021; Cesar et al., 2025; El-Afifi, Eladl, Sedhom, & Hassan, 2025).

1.3. Contributions

Considering all of the above, the contributions of this article are listed below:

1. Analysis of near-instantaneous prognosis when discharge current (exogenous inputs) is characterized as autoregressive models.

2. Feasibility confirmation of applying near-instantaneous prognosis in systems whose exogenous inputs are represented by autoregressive models that comply with the strict stationarity property.

3. Development of quasi-instantaneous prognosis when the strict stationarity property is no longer fulfilled by exogenous inputs.

The continuity of this approach is key to make further steps towards closing control loops that feed back prognostic information, endowing energy-aware control systems with an anticipatory capacity in a formal fashion.

1.4. Structure of the article

This article is structured as follows. In Section 2, background is presented regarding the problem that has just been raised. It briefly explains how prediction of events has been formalized by means of a probability measure for the time of occurrence of a future event. The concept of *"near-instantaneous prognosis"* is presented below as it was originally introduced. Later, in Section 3, the problem is formulated in the context of EoD time prognosis. In Section 4, it is shown that when these strictly stationary autoregressive models are used as exogenous inputs, near-instantaneous prognosis can be achieved. In addition, we present a new, more general methodology called *"quasi-instantaneous prognosis"*, which does not require stationarity conditions on exogenous inputs. Finally, Section 5 outlines the conclusions of this work. For the sake of completeness, we review autoregressive models and study the conditions for strict stationarity in Appendix A.

2. BACKGROUND

There are some elements that need to be defined in order to present the aforementioned contributions in a self-contained way. In Section 2.2 We begin by establishing a formal theoretical framework for the event prognosis problem in battery monitoring, which involves defining a precise mathematical formulation to compute the probability that the event *'End-of-Discharge'* will occur at a given future time

Since future inputs are unknown (discharge current), these must be modeled in order to prognosticate. Section 2.3 serves as an introduction to near-instantaneous prognosis under its original formulation, which considers stationarity conditions on the exogenous inputs. Prior to this introduction, some definitions of stationarity are provided to facilitate understanding of what comes next.

2.1. Philosophical discussion

Philosophically, how can a prediction be corroborated if the future does not exist? To a large extent, current prognostic algorithms based on machine learning aim to identify correlations between present patterns—or within a historical time window—and future events that have occurred under similar conditions. However, it could be unfair to evaluate a decision based on what is known to date and also on what will happen in the future, when at the time of deciding the future is unknown. The most formal alternative would be to accept that there is uncertainty about the future, and focus on properly characterizing and quantifying it through a causal dynamical model. However, it must be acknowledged that if quality data is available, even under a more informal approach (meta-heuristic models, for example), an algorithm that finds correlations can be very effective in practice.

Taking the above discussion into consideration, in this article we want to approach prognosis from the following perspective. Consider a discrete-time, non-linear dynamical system for a battery under a state-space representation

$$x_{k+1} = f(x_k, u_k, \omega_k) \tag{1}$$
$$y_k = g(x_k, u_k, \eta_k), \tag{2}$$

where x_k is the State-of-Charge (SoC) of a battery, y_k is voltage measured across terminals of the battery, u_k denotes the discharge current considered as exogenous input, and ω_k and η_k are process and measurement noise respectively. The objective is to predict the occurrence time of the future event *"End-of-Discharge"* for this dynamical system in an effective (i.e. with great accuracy) and efficient (i.e. that takes the least amount of time possible) way. Moreover, we would like to provide mathematical guarantees of convergence to the correct results in some sense.

The mathematical foundation for formally defining efficacy has been established in a recently published work, where the concept of *'near-instantaneous prognosis'* is introduced and successfully applied to predicting the End-of-Discharge (EoD) time in batteries (Acuña-Ureta & Orchard, 2022). This concept refers to prognostic algorithms with maximal performance, meaning they achieve mathematical convergence to correct results while requiring minimal computational resources. As a result, they can be implemented on cheap and low-performance microprocessors. This is attainable under the condition that the exogenous input is required to converge to a strictly stationary stochastic process.

2.2. Theory of Uncertain Event Prognosis

Before delving into the probabilistic assessment of the timing of a forthcoming event (Acuña-Ureta et al., 2021), it's essential to establish some groundwork:

1. We define a stochastic process $\{X_k, U_k\}_{k\in\mathbb{N}}$ representing the uncertain evolution of the battery SoC (X_k), current discharge (U_k), where $k \in \mathbb{N}$ indexes time.
2. $\mathcal{E}$ =*"End-of-Discharge"* denotes the event of interest, which is EoD.
3. $v_{cut-off}$ represents a voltage threshold, the crossing of

which triggers the event $\mathcal{E}$.

Note that triggering the EoD event when a battery voltage drops and crosses a threshold is very common. Typically, battery-powered devices (computers, phones, etc.) automatically shut down when this situation occurs.

Given these definitions and indicating k_p as the present time, the time of occurrence of $\mathcal{E}$ is defined as (Daigle & Goebel, 2013):

$$\tau_{\mathcal{E}}(k_p) := \inf\left\{k \in \mathbb{N} : \{k > k_p\} \wedge \{\text{EoD at time } k\}\right\}. \quad (3)$$

Since the event occurrence depends on the stochastic process $\{X_k, U_k\}_{k\in\mathbb{N}}$, $\tau_{\mathcal{E}}$ is a random variable. Logically, we must define the event we aim to predict so that we can derive a probability distribution for $\tau_{\mathcal{E}}$.

At each time step k, $\mathcal{E}$ may or may not occur, each with a certain probability. We can define a binary stochastic process $\{E_k\}_{k\in\mathbb{N}}$, such that, for each $k \in \mathbb{N}$, $E_k = e_k \in \{\mathcal{E}, \mathcal{E}^c\}$ and:

$$\mathbb{P}(E_k = \mathcal{E}^c) = 1 - \mathbb{P}(E_k = \mathcal{E}), \quad (4)$$

where $\mathcal{E}^c$ denotes the non-occurrence of $\mathcal{E}$. The goal is to find the first occurrence in time of $\mathcal{E}$, denoted as $\tau_{\mathcal{E}} = \tau_{\mathcal{E}}(k_p)$, which can be redefined in terms of $\{E_k\}_{k\in\mathbb{N}}$ as

$$\tau_{\mathcal{E}}(k_p) := \inf\left\{k \in \mathbb{N} : \{k > k_p\} \wedge \{E_k = \mathcal{E}\}\right\}. \quad (5)$$

Schematically, the statistical dependence between these variables is illustrated in Fig. 1. At each time point k, the event is declared or not, depending exclusively on the SoC and discharge current at that time point and not at other times. Furthermore, to construct $\tau_{\mathcal{E}}$, all event declarations are included, and $\tau_{\mathcal{E}}$ is defined as the time at which it was first declared.

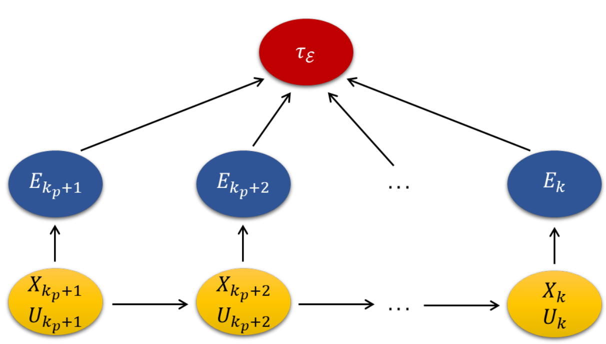


Figure 1. Statistical dependence relationship between variables (Acuña-Ureta & Orchard, 2022).

According to (Acuña-Ureta & Orchard, 2022), the probability distribution of $\tau_{\mathcal{E}}$ is given by:

$$\mathbb{P}(\tau_{\mathcal{E}} = k) := \int_{\mathbb{X}_{k_p+1:k}} \int_{\mathbb{U}_{k_p+1:k}} \mathbb{P}\left(E_k = \mathcal{E} | x_k, u_k\right) \ldots \prod_{j=k_p+1}^{k-1} \left[1 - \mathbb{P}\left(E_j = \mathcal{E} | x_j, u_j\right)\right] \ldots p(x_{k_p+1:k}, u_{k_p+1:k}) du_{k_p+1:k} dx_{k_p+1:k}. \quad (6)$$

In essence, this expression suggests that the probability of the event EoD occurring for the first time at a future time k, $k > k_p$, is computed by averaging all possible future SoC trajectories $x_{k_p+1:k}$ and discharge current trajectories $u_{k_p+1:k}$, evaluating the likelihood of EoD occurrence at time k, and non-previous occurrence.

2.3. Near-instantaneous prognosis

A key requirement for applying near-instantaneous prognosis (Acuña-Ureta & Orchard, 2022) is that the exogenous inputs must satisfy the property

$$p(u_{k_p+1:k}) = p(u_{k_p+1:k-1}) p(u_k). \quad (7)$$

Recall that, according to Eqs. (1)-(2), u_k denotes the battery discharge current. This property is achieved under the condition of strict-sense stationarity.

Definition 1 (Strict-sense stationarity) *Let $\{U_k\}_{k\in\mathbb{N}}$ be a stochastic process. It is said to be strict-sense or strongly stationary if the joint probability distribution of*

$$(U_k, U_{k+1}, \ldots, U_{k+n})$$

is the same as the joint probability distribution of

$$(U_{k+\tau}, U_{k+\tau+1}, \ldots, U_{k+\tau+n}),$$

for all $k, n, \tau \in \mathbb{N} \cup \{0\}$.

The near-instantaneous prognosis method consists of computing Eq. (6) very efficiently, and is as follows. First of all, note that Eq. (1) states that x_k does not depend on u_k, but rather on u_{k-1}. Therefore, we can manipulate the expression in Eq. (6) integrating with respect of u_k separately, as shown below

$$\mathbb{P}(\tau_{\mathcal{E}} = k) := \int_{\mathbb{X}_{k_p+1:k}} \int_{\mathbb{U}_{k_p+1:k}} \int_{\mathbb{U}_k} \mathbb{P}\left(E_k = \mathcal{E} | x_k, u_k\right) \ldots \prod_{j=k_p+1}^{k-1} \left[1 - \mathbb{P}\left(E_j = \mathcal{E} | x_j, u_j\right)\right] \ldots p(x_{k_p+1:k}, u_{k_p+1:k-1}) p(u_k) du_k du_{k_p+1:k-1} dx_{k_p+1:k} \quad (8)$$

$$= \int_{\mathbb{X}_{k_p+1:k}} \int_{\mathbb{U}_{k_p+1:k-1}} \mathbb{E}_{U_k} \{\mathbb{P}(E_k = \mathcal{E}|x_k, U_k)\}$$
$$\dots \prod_{j=k_p+1}^{k-1} \Big[1 - \mathbb{P}(E_j = \mathcal{E}|x_j, u_j)\Big]$$
$$\dots p(x_{k_p+1:k}, u_{k_p+1:k-1}) du_{k_p+1:k-1} dx_{k_p+1:k}. \quad (9)$$

Note that

$$\mathbb{E}_{U_k}\{\mathbb{P}(E_k = \mathcal{E}|x_k, U_k)\} = \int_{\mathbb{U}_k} \mathbb{P}(E_k = \mathcal{E}|x_k, u_k)\, p(u_k) du_k. \quad (10)$$

Let us draw N i.i.d. samples from the joint probability density $p(x_{k_p+1:k}, u_{k_p+1:k-1})$ (exogenous inputs are considered up to time $k-1$). That is, $\forall i \in \{1, \dots, N\}$,

$$\left(x^{(i)}_{k_p+1:k}, u^{(i)}_{k_p+1:k-1}\right) \sim p(x_{k_p+1:k}, u_{k_p+1:k-1}), \quad (11)$$

where $x^{(i)}_{k_p+1:k}$ and $u^{(i)}_{k_p+1:k-1}$ are the i^{th} simulated trajectories of future SoC and discharge current values, respectively. Note that

$$x^{(i)}_{k_p+1:k} := \{x^{(i)}_{k_p+1}, x^{(i)}_{k_p+2}, \dots, x^{(i)}_{k}\} \quad (12)$$
$$u^{(i)}_{k_p+1:k-1} := \{u^{(i)}_{k_p+1}, u^{(i)}_{k_p+2}, \dots, u^{(i)}_{k-1}\}. \quad (13)$$

Thus, $p(x_{k_p+1:k}, u_{k_p+1:k-1})$ can be weakly approximated by a sum of Dirac delta functions as

$$p(x_{k_p+1:k}, u_{k_p+1:k-1})$$
$$\approx \frac{1}{N} \sum_{i=1}^{N} \delta_{x^{(i)}_{k_p+1:k}, u^{(i)}_{k_p+1:k-1}}(x_{k_p+1:k}, u_{k_p+1:k-1}). \quad (14)$$

By replacing this approximation of $p(x_{k_p+1:k}, u_{k_p+1:k-1})$ in Eq. (9), we can alternatively define

$$\hat{\mathbb{P}}_N(\tau_{\mathcal{E}} = k) := \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}_{U_k}\left\{\mathbb{P}\left(E_k = \mathcal{E}|{x_k}^{(i)}, U_k\right)\right\}$$
$$\dots \prod_{j=k_p+1}^{k-1} \left[1 - \mathbb{P}\left(E_j = \mathcal{E}|{x_j}^{(i)}, {u_j}^{(i)}\right)\right], \quad (15)$$

so that $\mathbb{P}(\tau_{\mathcal{E}} = k) \approx \hat{\mathbb{P}}_N(\tau_{\mathcal{E}} = k)$, and, due to the Law of Large Numbers,

$$\mathbb{P}(\tau_{\mathcal{E}} = k) = \lim_{N \to +\infty} \hat{\mathbb{P}}_N(\tau_{\mathcal{E}} = k). \quad (16)$$

At first glance, the described method could be seen as way to implement Monte Carlo simulations. However, the advantage of formulating the approximation in this manner, by including the expected value $\mathbb{E}_{U_k}\left\{\mathbb{P}\left(E_k = \mathcal{E}|{x_k}^{(i)}, U_k\right)\right\}$ computed analytically, is that it is the overriding term in the expression, so a precise calculation of it makes $\hat{\mathbb{P}}_N(\tau_{\mathcal{E}} = \cdot)$ rapidly converge to $\mathbb{P}(\tau_{\mathcal{E}} = \cdot)$. Furthermore, the convergence is almost instantaneous, since according to (Acuña-Ureta & Orchard, 2022) excellent results are achieved when $N = 1$. For this reason it is said that the prognosis is "near-instantaneous."

3. END-OF-DISCHARGE TIME PROBABILITY WITH NEAR-INSTANTANEOUS PROGNOSIS

This section aims to show the structure of the dynamic model assumed to govern the evolution of a battery's SoC. Based on this model, the corresponding elements required for implementing a near-instantaneous prognosis scheme for predicting End-of-Discharge (EoD) time are subsequently derived.

3.1. State-of-Charge dynamical system

According to (Acuña-Ureta & Orchard, 2022) and Fig. 2, the battery state-space dynamics are defined by

$$x_{k+1} = x_k - v_{oc}(x_k) u_k \frac{T_s}{E_{crit}} + \omega_k \quad (17)$$
$$y_k = v_{oc}(x_k) - R u_k + \eta_k, \quad (18)$$

where x_k is the battery SoC, understood as a fraction of the battery's total capacity (E_{crit}). The parameter T_s is the sampling time, and ω_k is process noise, assumed to be additive and Gaussian with zero mean. In turn, y_k is the output voltage across battery terminals (shown as v in Fig. 2), R is the sum of the internal and polarization resistances assumed constant and known, and η_k is measurement noise, assumed additive, Gaussian with zero mean.

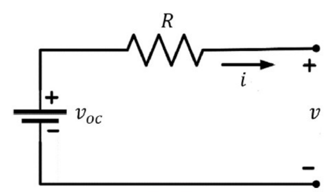


Figure 2. Simplified model of a battery (Acuña-Ureta & Orchard, 2022).

The open circuit voltage v_{oc} is a function of the battery SoC:

$$v_{oc}(x_k) = v_L + (v_0 - v_L) e^{\gamma(x_k - 1)} + \alpha v_L (x_k - 1)$$
$$\dots + (1 - \alpha) v_L (e^{-\beta} - e^{-\beta\sqrt{x_k}}). \quad (19)$$

The parameters $v_0, v_L, \alpha, \beta, \gamma$ can be estimated according to the method proposed in (Pola et al., 2015).

It's worth noting that Eq. (17) defines state transition and corresponds to Eq. (1). Similarly, Eq. (18) is the measurement equation responding to Eq. (2). Note that these equations describe a simplified representation and does not capture effects like hysteresis or relaxation seen in modern batteries.

3.2. End-of-Discharge event likelihood function

The event $\mathcal{E}$ denoting the *"End-of-Discharge"* of a battery, is assumed to be triggered whenever the output voltage y_k decreases and crosses a threshold, a cut-off voltage, denoted as $v_{cut-off}$. For our case study, and considering y_k as a random variable Y_k, the probability that the event happens at a time k conditional on specific SoC and discharge current values, can be expressed mathematically as

$$\mathbb{P}(E_k = \mathcal{E}|x_k, u_k) = \mathbb{P}(Y_k \leq v_{cut-off}|x_k, u_k). \quad (20)$$

Replacing Eq. (18) in Eq. (20) and considering measurement noise η_k as a random variable H_k, we get:

$$\begin{aligned}\mathbb{P}(E_k &= \mathcal{E}|x_k, u_k) \\ &= \mathbb{P}(v_{oc}(x_k) \leq v_{cut-off} + Ru_k + H_k|x_k, u_k) \quad (21)\\ &= \mathbb{P}(H_k \geq v_{oc}(x_k) - (v_{cut-off} + Ru_k)|x_k, u_k). \quad (22)\end{aligned}$$

Considering $\mu_k(u_k) = v_{cut-off} + Ru_k$, Eq. (22) becomes

$$\begin{aligned}\mathbb{P}(E_k &= \mathcal{E}|x_k, u_k) \\ &= \mathbb{P}\left(Z \geq \left(\frac{v_{oc}(x_k) - \mu_k(u_k)}{\sigma_\eta}\right)\middle| x_k, u_k\right) \quad (23)\\ &= 1 - \mathbb{P}\left(Z < \left(\frac{v_{oc}(x_k) - \mu_k(u_k)}{\sigma_\eta}\right)\middle| x_k, u_k\right) \quad (24)\\ &= 1 - \Phi_{\mu_k(u_k),\sigma_\eta}\big(v_{oc}(x_k)\big). \quad (25)\end{aligned}$$

A couple of things worth noting. We have $H_k \sim \mathcal{N}(0, \sigma_\eta^2)$, and that is why $Z = \frac{H_k}{\sigma_\eta} \sim \mathcal{N}(0, 1)$. With this in mind, $\Phi_{\mu_k(u_k),\sigma_\eta}(\,\cdot\,)$ is a Gaussian cumulative distribution function with mean $\mu_k(u_k) = v_{cut-off} + Ru_k$ and standard deviation σ_η.

3.3. End-of-Discharge time probability distribution estimation

Replacing Eq. (25) in Eq. (9), we get

$$\begin{aligned}\mathbb{P}(\tau_{\mathcal{E}} = k) := \int_{\mathbb{X}_{k_p+1:k}} \int_{\mathbb{U}_{k_p+1:k}} \Big[1 - \Phi_{\mu_k(u_k),\sigma_\eta}\big(v_{oc}(x_k)\big)\Big] \\ \ldots \prod_{j=k_p+1}^{k-1} \Phi_{\mu_j(u_j),\sigma_\eta}\big(v_{oc}(x_j)\big) \\ \ldots p(x_{k_p+1:k}, u_{k_p+1:k}) du_{k_p+1:k} dx_{k_p+1:k}. \quad (26)\end{aligned}$$

Remembering Eq. (7), we can express Eq. (26) as

$$\begin{aligned}&\mathbb{P}(\tau_{\mathcal{E}} = k) \\ &:= \int_{\mathbb{X}_{k_p+1:k}} \int_{\mathbb{U}_{k_p+1:k-1}} \left(\int_{\mathbb{U}_k} \Big[1 - \Phi_{\mu_k(u_k),\sigma_\eta}\big(v_{oc}(x_k)\big)\Big] \right.\\ &\left. \ldots p(u_k) du_k \right) \prod_{j=k_p+1}^{k-1} \Phi_{\mu_j(u_j),\sigma_\eta}\big(v_{oc}(x_j)\big) \\ &\ldots p(x_{k_p+1:k}, u_{k_p+1:k-1}) du_{k_p+1:k-1} dx_{k_p+1:k}. \quad (27)\end{aligned}$$

Noting that

$$\begin{aligned}\int_{\mathbb{U}_k} \Big[1 - \Phi_{\mu_k(u_k),\sigma_\eta}\big(v_{oc}(x_k)\big)\Big] p(u_k) du_k \\ = 1 - \mathbb{E}_{U_k}\left\{\Phi_{\mu_k(U_k),\sigma_\eta}\big(v_{oc}(x_k)\big)\right\}, \quad (28)\end{aligned}$$

and by replacing this term, we get the following equation, a cornerstone for this article:

$$\begin{aligned}&\mathbb{P}(\tau_{\mathcal{E}} = k) \\ &:= \int_{\mathbb{X}_{k_p+1:k}} \int_{\mathbb{U}_{k_p+1:k-1}} \Big[1 - \mathbb{E}_{U_k}\left\{\Phi_{\mu_k(U_k),\sigma_\eta}\big(v_{oc}(x_k)\big)\right\}\Big] \\ &\ldots \prod_{j=k_p+1}^{k-1} \Phi_{\mu_j(u_j),\sigma_\eta}(v_{oc}(x_j)) \\ &\ldots p(x_{k_p+1:k}, u_{k_p+1:k-1}) du_{k_p+1:k-1} dx_{k_p+1:k}. \quad (29)\end{aligned}$$

Thus, the key for near-instantaneous prognosis is to analytically calculate the following term

$$\mathbb{E}_{U_k}\left\{\Phi_{\mu_k(U_k),\sigma_\eta}\big(v_{oc}(x_k)\big)\right\}, \quad (30)$$

that depends in turn on the statistics of the exogenous input u_k, assumed to be characterized by autoregressive models in this article, which are revisited in the next section.

As explained in Section 2.3, and analogously to Eq. (15), near-instantaneous prognosis of the EoD time probability distribution is given by

$$\begin{aligned}\hat{\mathbb{P}}_N(\tau_{\mathcal{E}} = k) := \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}_{U_k}\left\{\Phi_{\mu_k(U_k),\sigma_\eta}\left(v_{oc}\left({x_k}^{(i)}\right)\right)\right\} \\ \ldots \prod_{j=k_p+1}^{k-1} \Phi_{\mu_j\left({u_j}^{(i)}\right),\sigma_\eta}\left(v_{oc}\left({x_j}^{(i)}\right)\right). \quad (31)\end{aligned}$$

4. PROGNOSIS WITH (NON-)STATIONARY AUTOREGRESSIVE EXOGENOUS INPUTS

In (Acuña-Ureta & Orchard, 2022) it was shown that when the exogenous input describes a strict-sense stationary stochastic process, then it is feasible to implement near-instantaneous prognosis. In the same work, the effectiveness of the method was shown in an example where the exogenous input is characterized by a two-state Markov Chain, which is known to converge to a stationary probability distribution, so the condition is asymptotically met. Since this paper studies characterizations using autoregressive models, this section begins by empirically demonstrating the effectiveness of the near-instantaneous prognosis method when the strict-sense stationarity condition is met for the battery discharge current model. However, we go a step further, which is the main contribution of this article. We present a generalization of the near-instantaneous prognosis method where the exogenous input model, i.e., the model describing the dynamics of the exogenous input, does not satisfy the strict-sense stationarity condition, which we call quasi-instantaneous prognosis.

As we will see in Section 4.1, near-instantaneous prognosis requires only one simulation to achieve convergence. Since its performance is maximal, theoretically no simulation-based prognostic method can be more efficient (although it might be more effective), hence its name. On the other hand, Section 4.2 presents quasi-instantaneous prognosis, which generalizes the previous method but requires a greater number of simulations (more than ten to obtain reasonable results). Nevertheless, it is still efficient enough to be used in embedded systems for real-time applications, which is why its name retains the concept of "instantaneous."

4.1. Near-instantaneous prognosis: Stationary input

Among the autoregressive models discussed in Section A.1, the only feasible ones that can meet the stationarity condition in the strict sense are the AR, MA, and ARMA models. Both AR and MA models are particular cases of ARMA models. Consequently, an example of an ARMA model is shown below to illustrate how satisfying the strict stationarity condition effectively makes it possible to implement near-instantaneous prognosis. Indeed, in previous sections we described an autoregressive moving average model as

$$u_k = c + \sum_{i=1}^{p} \phi_i u_{k-i} + \sum_{j=1}^{q} \theta_j r_{k-j} + r_k,\ k \in \mathbb{N}. \tag{32}$$

Note that a drift parameter c has been included, which makes this expression slightly different from Eq. (54). This parameter does not affect the strict-sense stationarity condition, it only shifts the mean of u_k.

If this models the discharge current of a battery, and is such that is a strict-sense stationary process and $r_k \sim \mathcal{N}\left(0, {\sigma_r}^2\right)$, i.e. conditions in Section A.2 are met, then

$$\Rightarrow u_k \sim \mathcal{N}\left(\frac{c}{1-\sum_{i=1}^{p}\phi_i}, {\sigma_u}^2\right), \tag{33}$$

where ${\sigma_u}^2$ is a fixed variance to which the stochastic process numerically converges.

Considering a known strict-sense stationary ARMA model, then Eq. (30) now yields a close expression:

$$\mathbb{E}_{U_k}\left\{\Phi_{\mu_k(u_k),\sigma_\eta}\left(v_{oc}(x_k)\right)\right\} = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{v_{oc}\left(x_k\right) - \left(v_{\text{cut off}} + R\mu_u\right)}{\sqrt{2(\sigma_\eta^2 + R^2{\sigma_u}^2)}}\right)\right], \tag{34}$$

where $\mu_u = \frac{c}{1-\sum_{i=1}^{p}\phi_i}$.

For example, consider the following ARMA(2,2) model that characterize a discharge current profile:

$$u_k = 5.12 - 0.3u_{k-1} + 0.02u_{k-2} + r_k - 0.2r_{k-1} + 0.01r_{k-2}, \tag{35}$$

with $r_k \sim \mathcal{N}(0, 0.5^2)$. It satisfies stationary conditions in strict-sense.

If we neglect the drift term $c = 5.12$, we have

$$U(z) = \frac{\Theta(z)}{\Phi(z)}R(z) \tag{36}$$

$$= \frac{0.01z^{-2} - 0.2z^{-1} + 1}{-0.02z^{-2} + 0.3z^{-1} + 1}R(z) \tag{37}$$

$$= \frac{(z-0.100)^2}{(z+0.356)(z-0.056)}R(z), \tag{38}$$

therefore, all the singularities of $\frac{\Theta(z)}{\Phi(z)}$ are within the unit circle of the complex plane, and $\{U_k\}_{k>k_p}$ is stationary in strict-sense. A realization of this stochastic process is shown in Fig. 3, where roughly $u_k \sim \mathcal{N}\left(4.00, 0.57^2\right)$.

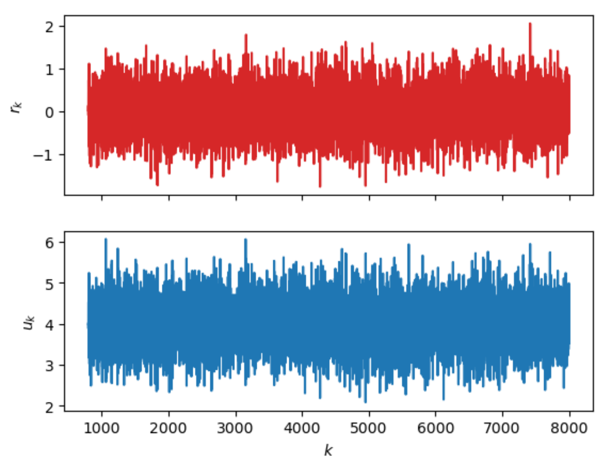


Figure 3. Single realization of a strictly stationary stochastic process ARMA(2,2) depicted in Eq. (35).

By implementing near-instantaneous prognosis with Eq. (31), a comparison can be made with results that would be obtained using the traditional method of Monte Carlo simulations to solve Eq. (29), as shown in Fig. 4.

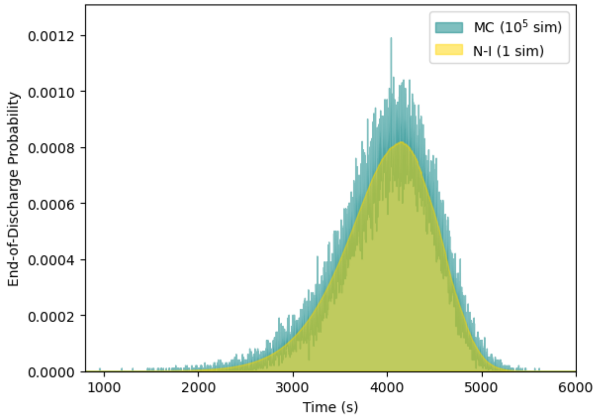


Figure 4. Comparison of future EoD time probability distribution approximated using: i) Traditional Monte Carlo (MC) with $N = 100.000$ simulations, and ii) Near-instantaneous (N-I) prognosis with $N = 1$ simulation.

As can be seen in Fig. 4, a single simulated trajectory of future inputs (discharge current) is sufficient for the near-instantaneous prognosis method to achieve immediate convergence. A stark contrast can be seen when compared with the results associated with the traditional Monte Carlo method, where even with 100.000 trajectories it is clear that more simulations are needed for the result to converge.

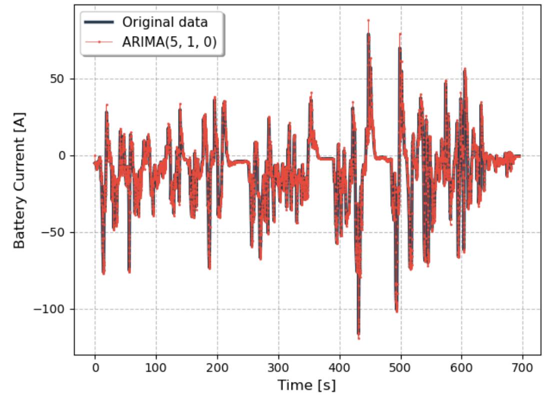
4.2. Quasi-instantaneous prognosis: Non-stationary input

In Section 2.3, we saw that the condition of Eq. (7) is necessary to use the near-instantaneous prognosis method, which holds when $\{U_k\}_{k>k_p}$ is a strict-sense stationary stochastic process. But what can be done if this is not met, for example when $\{U_k\}_{k>k_p}$ is characterized by an ARIMA model? ARIMA models are non-stationary by construction. A very illustrative example where we can see this is in electromobility, as shown in Figure 5. It depicts actual current profiles from a BMW i3 (60 Ah) electric vehicle (EV). Specifically, they correspond to profiles A14 and B07 from the public dataset *Battery and Heating Data in Real Driving Cycles* available on IEEE*DataPort* (Steinstraeter, Buberger, & Trifonov, 2020). The ARIMA models were fitted using the `pmdarima` Python library, minimizing the Akaike Information Criterion (AIC). This article proposes a variant of the near-instantaneous prognosis method for this type of situations: quasi-instantanous prognosis.

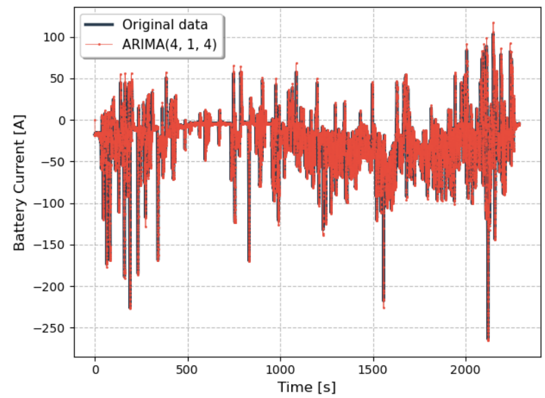
Note that, in general, it is always true that

$$p(u_{k_p+1:k}) = p(u_{k_p+1:k-1})p(u_k|u_{k_p+1:k-1}). \tag{39}$$

Therefore, under this idea, in quasi-instantanous prognosis Eq. (9) would become



(a) Summer season (Measurement A14).



(b) Winter season (Measurement B07).

Figure 5. Real-world current profiles of a BMW i3 electric vehicle (Steinstraeter et al., 2020). Profiles are fitted with optimal ARIMA models that minimize the Akaike Information Criterion (AIC). The variance of the residual error in Figure 5a is 0,519784, while in Figure 5b it is 1,464339.

$$\begin{aligned}
&\mathbb{P}(\tau_{\mathcal{E}} = k) \\
&\quad = \int_{\mathbb{X}_{k_p+1:k}} \int_{\mathbb{U}_{k_p+1:k-1}} \mathbb{E}_{U_k|U_{k_p+1:k-1}} \left\{ \mathbb{P}\left(E_k = \mathcal{E} | x_k, U_k\right)\right\} \\
&\quad \ldots \prod_{j=k_p+1}^{k-1} \left[1 - \mathbb{P}\left(E_j = \mathcal{E}|x_j, u_j\right)\right] \\
&\quad \ldots p(x_{k_p+1:k}, u_{k_p+1:k-1}) du_{k_p+1:k-1} dx_{k_p+1:k},
\end{aligned} \tag{40}$$

which can be approximated as

$$\begin{aligned}
&\hat{\mathbb{P}}_N(\tau_{\mathcal{E}} = k) \\
&\qquad := \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}_{U_k|u^{(i)}_{k_p+1:k-1}} \left\{ \mathbb{P}\left(E_k = \mathcal{E}|x_k{}^{(i)}, U_k\right)\right\} \\
&\qquad \ldots \prod_{j=k_p+1}^{k-1} \left[1 - \mathbb{P}\left(E_j = \mathcal{E}|x_j{}^{(i)}, u_j{}^{(i)}\right)\right],
\end{aligned} \tag{41}$$

where the statistics of U_k are conditional on an inputs trajectory $u_{k_p+1:k-1}$. What has changed is that it is now required to compute the conditional expectation

$$\mathbb{E}_{U_k|u_{k_p+1:k-1}}\left\{\mathbb{P}\left(E_k=\mathcal{E}|x_k,U_k\right)\right\}. \tag{42}$$

Let's return to the EoD time prediction case study. Then, Eq. (41) becomes

$$\begin{aligned}
&\hat{\mathbb{P}}_N(\tau_{\mathcal{E}}=k)\\
&\quad=\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}_{U_k|u^{(i)}_{k_p+1:k-1}}\left\{\Phi_{\mu_k(U_k),\sigma_\eta}\left(v_{oc}\left({x_k}^{(i)}\right)\right)\right\}\\
&\quad\cdots\prod_{j=k_p+1}^{k-1}\Phi_{\mu_j\left({u_j}^{(i)}\right),\sigma_\eta}\left(v_{oc}\left({x_j}^{(i)}\right)\right).
\end{aligned} \tag{43}$$

Let's analyze a numerical example to study the impact of assuming a non-stationary exogenous input. Assume that a battery's input, which is the discharge current, is described by the following ARIMA(1,1,1) model:

$$(1-0.3z^{-1})(1-z^{-1})U(z)=(1+0.02z^{-1})R(z), \tag{44}$$

with $r_k\sim\mathcal{N}(0,0.01^2)$. Then,

$$u_k=1.3u_{k-1}-0.3u_{k-2}+r_k+0.02r_{k-1}. \tag{45}$$

This can also be understood as a non-stationary ARMA(2,2) model. Indeed,

$$\begin{aligned}
U(z)&=\frac{\Theta(z)}{\Phi(z)}R(z) &(46)\\
&=\frac{0.02z^{-1}+1}{0.03z^{-2}-1.3z^{-1}+1}R(z) &(47)\\
&=\frac{z(z+0.02)}{(z-1.277)(z-0.024)}R(z), &(48)
\end{aligned}$$

which we can observe denotes a pole outside the unit circle, so $\{U_k\}_{k>k_p}$ is non-stationary. A realization of this stochastic process is shown in Fig. 6.

By implementing quasi-instantaneous prognosis (adapted for non-stationary inputs) with Eq. (43), a comparison can be made with results that would be obtained using the traditional method of Monte Carlo simulations to solve Eq. (40), as shown in Fig. 7.

Unlike when the input was characterized by a strict-sense stationary autoregressive model, there are clearly differences in the number of simulations required to achieve convergence. In the first case (stationary, Section 4.1), near-instantaneous prognosis was surprisingly efficient, converging in just one simulation, which is the minimum effort that can be made when analyzing the future. In other words, convergence was immediate with minimal effort, implying maximal performance.

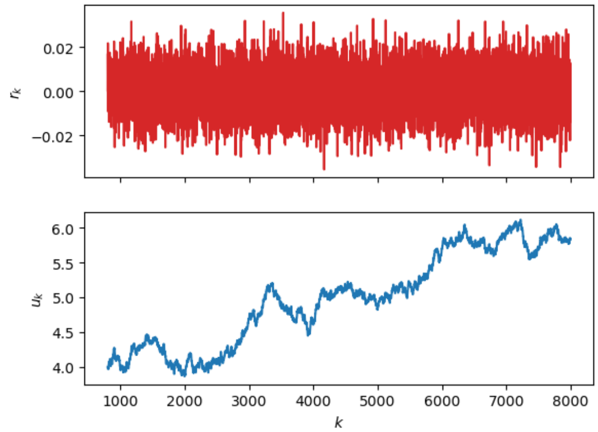


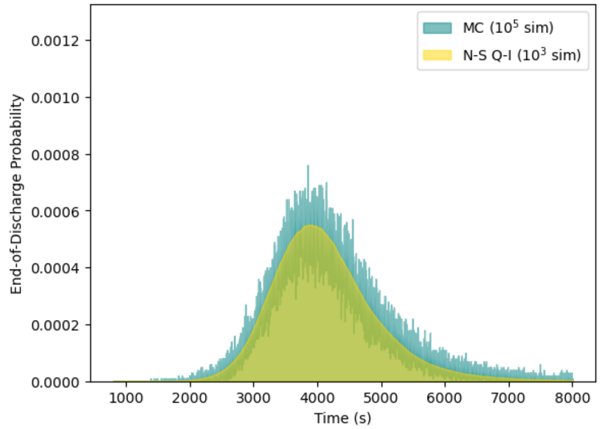
Figure 6. Single realization of a non-stationary stochastic process ARMA(1,1,1) depicted in Eq. (45).

In contrast, in the second case (non-stationary, Section 4.2) quasi-instantaneous prognosis shows inferior performance, requiring more than 100 simulations to achieve convergence although reasonable approximations (depending on the application) might be obtained with 10 simulations. However, the methodology generalized to non-stationary cases presented in this section still shows promising results compared to the traditional Monte Carlo simulation method, being superior by several orders of magnitude in terms of computational resources.

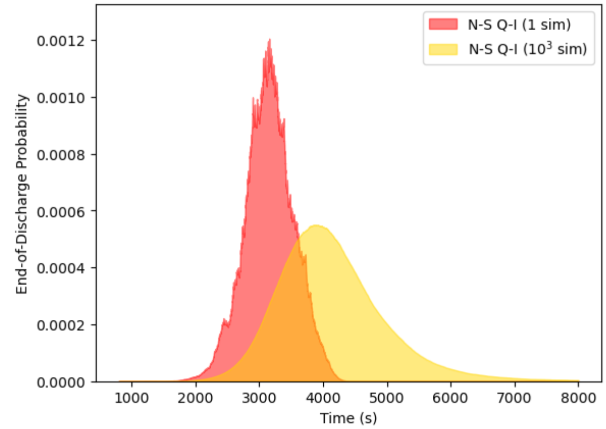
4.3. Advantages of quasi-instantaneous prognosis

In Section 1.1, we reviewed different methods for EoD prognosis, including those that seek to establish correlations using data-driven approaches, and others that aim to propagate uncertainty through statistical models or those based on Bayesian filters (such as Particle Filters and variants of the Kalman Filter). Unlike these methods, quasi-instantaneous prognosis offers the following advantages:

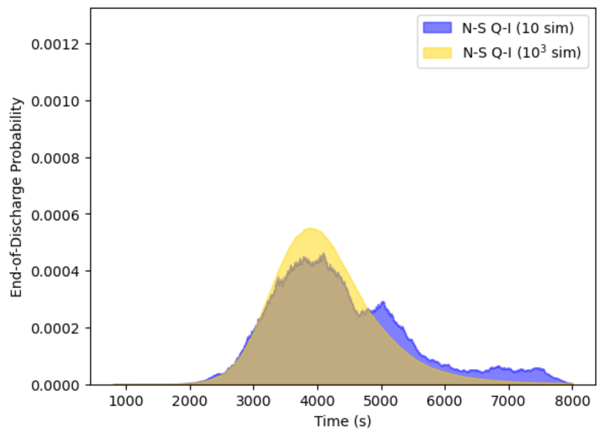
- Quasi-instantaneous prognosis is a method based on causality provided by a dynamic model. This article uses a simplified battery model, but it could easily be replaced with a more complex one.
- It leverages the analytical probability measure for prognosticating uncertain future events within a formal framework recently published in the literature (Acuña-Ureta et al., 2021). Prior to this, it was necessary to approximate probabilities with Monte Carlo simulations, or if the probability was known, it was only applicable to a very limited family of dynamic models.
- The formal framework ensures that quasi-instantaneous prognosis converges stochastically with increasing the number of simulations. Convergence is a crucial concept, but previous work in the prognostic literature has



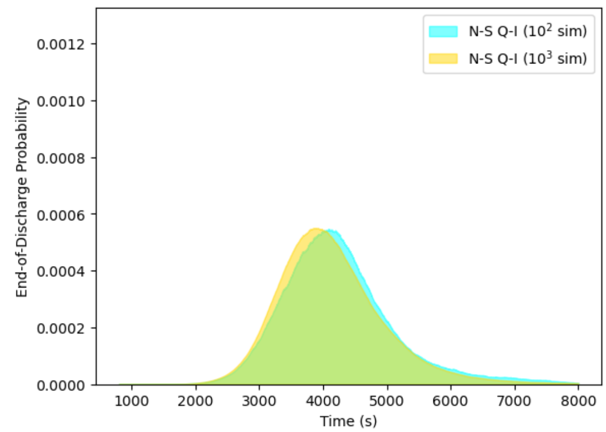
(a) Level of convergence using Monte Carlo simulations with 100.000 simulations.



(b) Level of convergence using quasi-instantaneous prognosis with 1 simulation.



(c) Level of convergence using quasi-instantaneous prognosis with 10 simulation.



(d) Level of convergence using quasi-instantaneous prognosis with 100 simulation.

Figure 7. Comparison levels of convergence of future EoD time probability distribution approximated using traditional Monte Carlo (MC) simulations and Quasi-Instantaneous (Q-I) prognosis with $N \in \{1, 10, 100, 1000\}$ simulations. Future inputs are characterized by the ARIMA(1,1,1) model expressed in Eq. (45). Since Q-I prognosis with $N = 1000$ has converged already, it is therefore used as benchmark.

addressed it rather empirically -as far as we know- due to the lack of a formal prognostic framework.

- The proposed method is highly efficient, even suitable for embedded processing, and does not require assuming that the discharge profile describes the properties of a stationary stochastic process.

Other analytical methods exist based on concepts such as *First-Passage Time* (FPT) (Siegert, 1951) and *First-Hitting Time* (FHT) (Salminen, 1988). These typically model physical phenomena, are formulated in continuous-time, and seek to characterize probabilistically the time at which a predetermined threshold is reached by a variable. The dynamics of this variable are modeled by a particular family of stochastic processes, such as a Poisson or gamma process, or alternatively, a diffusion process using Brownian motion (Deng, Barros, & Grall, 2014; Klump & Kolb, 2023; Abundo, 2025; Lefebvre, 2026). This is much more aligned with a reliability engineering framework where regularity prior statistical conditions are assumed. We could say that the dynamic discharge model of a battery in general is not limited to a particular family of stochastic processes; its statistics depend on the user profile, and it also occurs in discrete-time. In other words, it corresponds more to a condition monitoring context. In this article, we have proposed a method that allows for arbitrarily precise approximation and have opened the door to characterizing the current profile with any non-stationary stochastic process, making it much more flexible for a variety of applications.

5. CONCLUSION

In this article, we revisit the near-instantaneous prognosis method and present a new variant for predicting the End-of-Discharge

Criteria	Near-Instantaneous Prognosis	Quasi-Instantaneous Prognosis
Assumption	Strictly stationary exogenous inputs	Non-stationary exogenous inputs
Number of simulations for convergence	$N \approx 1$	$N \approx 100$
Number of simulations for a reasonable approximation	$N \approx 1$	$N \approx 10$

Table 1. Comparative summary between Near-Instantaneous (N-I) and Quasi-Instantaneous (Q-I) Prognosis. For reference, a 1000-step prediction simulation takes approximately 0.05 seconds with a processor Intel®Core™i7-8550U CPU @ 1.80GHz 1.99GHz. Therefore, under these conditions (1000-step prediction), N-I Prognosis takes 0.05 seconds ($N = 1$), while Q-I Prognosis takes approximately 0.5 seconds ($N = 10$) to achieve reasonable results in terms of statistical convergence. Both approaches can be implemented in embedded battery monitoring systems.

time of batteries, which is called “quasi-instantaneous prognosis.” Originally, the method was presented assuming that the exogenous input, corresponding to the discharge current, must meet the condition of being able to be modeled as a strict-sense stationary stochatic process, a condition that is met for a Markov Chain when it reaches its stationary probability distribution. This article explores the characterization of discharge current through autoregressive models. Within this family of models, the strict stationarity condition is met only under certain circumstances, which are reviewed in the article. It is therefore verified that near-instantaneous prognosis effectively works with maximal performance in scenarios with inputs modeled by autoregressive models that meet the strict-sense stationarity condition. A single simulation of the future is required to achieve convergence, which contrasts sharply with the millions of simulations required by traditional Monte Carlo simulations. However, there is still a wide range of applications in engineering where non-stationary autoregressive models, like ARIMA models, are systematically used to model signals, such as solar and wind resources, or energy demand in an electrical grid. One of the most relevant contributions of this article is to extend the concepts of near-instantaneous prognosis to these types of non-stationary current profile scenarios, leading to the new concept of quasi-instantaneous prognosis. Even though the performance in terms of computational resources and level of convergence in the probability distribution is inferior to the strict-sense stationary case, the generalized quasi-instantaneous prognosis method for non-stationary cases is still far superior to traditional Monte Carlo simulations, and also offers a guarantee of stochastic convergence.

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BIOGRAPHIES

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A. STATIONARITY OF AUTOREGRESSIVE MODELS

One of the main contributions of this article is the study of a near-instantaneous prognosis when the discharge current u_k is characterized by an autoregressive model, even when it is not stationary in a strict-sense. Next, we will briefly review some types of autoregressive models to later study the stationarity conditions that, according to the literature, would allow the implementation of near-instantaneous prognosis.

A.1. Autoregressive models

The mathematical notation with which the autoregressive models are denoted in what follows considers the use of the one-sided Z-Transform, being used regularly to denote time-shifts:

$$x_{k-n} = z^{-n} x_k, \text{ for } k, n \in \mathbb{N}. \tag{49}$$

Let also $r_k \sim \mathcal{N}(0, {\sigma_r}^2)$ be an error term for all $k \in \mathbb{N}$.

- *Auto-Regressive (AR) process:* An autoregressive stochastic process AR(p) can be expressed as

$$u_k = c + \sum_{i=1}^{p} \phi_i u_{k-i} + r_k, \; k \in \mathbb{N}, \tag{50}$$

  where $p \in \mathbb{N}$ denotes number of regressors, $\phi_i \in \mathbb{R}$ is a real-valued coefficient for all $i \in \{1, 2, \ldots, p\}$, and c is a drift term. This stochastic process can also be expressed as

$$\left(1 - \sum_{i=1}^{p} \phi_i z^{-i}\right) u_k = c + r_k, \; k \in \mathbb{N}, \tag{51}$$

  where $\Phi(z) = 1 - \sum_{i=1}^{p} \phi_i z^{-i}$ is known as its characteristic polynomial.

- *Moving Average (MA) process:* An autoregressive stochastic process MA(q) can be expressed as

$$u_k = \sum_{j=1}^{q} \theta_j r_{k-j} + r_k,\ k \in \mathbb{N}, \tag{52}$$

where $q \in \mathbb{N}$ denotes number of regressors, $\theta_j \in \mathbb{R}$ is a real-valued coefficient for all $j \in \{1, 2, \ldots, q\}$. This stochastic process can also be expressed as

$$u_k = \left(1 + \sum_{j=1}^{q} \theta_j z^{-j}\right) r_k,\ k \in \mathbb{N}, \tag{53}$$

where $\Theta(z) = 1 + \sum_{j=1}^{q} \theta_j z^{-j}$ is known as its characteristic polynomial.

- *Auto-Regressive Moving Average (ARMA) process:* An autoregressive stochastic process ARMA(p, q) is a combination between an AR(p) and a MA(q) process. It can be expressed as

$$u_k - \sum_{i=1}^{p} \phi_i u_{k-i} = \sum_{j=1}^{q} \theta_j r_{k-j} + r_k,\ k \in \mathbb{N}, \tag{54}$$

where $p, q \in \mathbb{N}$ denote numbers of regressors, and $\phi_i, \theta_j \in \mathbb{R}$ are real-valued coefficients for all $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, q\}$. This stochastic process can also be expressed as

$$\left(1 - \sum_{i=1}^{p} \phi_i z^{-i}\right) u_k = \left(1 + \sum_{j=1}^{q} \theta_j z^{-j}\right) r_k,\ k \in \mathbb{N}, \tag{55}$$

or alternatively,

$$\Phi(z) u_k = \Theta(z) r_k,\ k \in \mathbb{N}. \tag{56}$$

- *Auto-Regressive Integrated Moving Average (ARIMA) process:* An autoregressive stochastic process ARIMA(p, D, q) can be expressed as

$$\left(1 - \sum_{i=1}^{p} \phi_i z^{-i}\right) w_k = \left(1 + \sum_{j=1}^{q} \theta_j z^{-j}\right) r_k,\ k \in \mathbb{N}, \tag{57}$$

where

$$w_k = \left(1 - z^{-1}\right)^D u_k. \tag{58}$$

The parameters $p, q \in \mathbb{N}$ denote numbers of regressors, and $\phi_i, \theta_j \in \mathbb{R}$ are real-valued coefficients for all $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, q\}$. Note that somehow w_k is by definition the D-th order difference (analogous to derivative in continuous-time) of u_k. In this regard, an ARIMA(p, D, q) model can be interpreted as an ARMA($p + D, q$) model for u_k, with D unit roots. We'll see shortly that because of this, no ARIMA model can be a strictly stationary stochastic process for u_k (however it can be for w_k).

A.2. Strict-sense stationarity conditions

Near-instantaneous prognosis was originally introduced with the requirement that Eq. (7) must hold, which occurs when the exogenous input (discharge current) is stationary in the strict sense. Consequently, it is appropriate to study the stationarity of autoregressive models. We begin by analyzing the ARMA model, noting that both AR and MA models are particular cases of it. Subsequently, we examine the ARIMA model, which exhibits notable differences compared to the ARMA formulation.

A.2.1. Strict-sense stationarity of ARMA(p,q) model

Yohai and Maronna discuss in (Yohai & Maronna, 1977) a first condition related to expected values, condition on which Bougerol and Picard (Bougerol & Picard, 1992) construct a generalized non-anticipatory (causal) solution for ARMA(p, q) processes in matrix terms. In 2010, Brockwell and Lidner (Brockwell & Lindner, 2010) expanded Yohai and Marona's ideas to build strictly stationary ARMA(p, q) processes, establishing necessary and sufficient conditions. This work is later expanded in (Vollenbröker, 2011). The conditions for strict-sense stationary solutions for ARMA(p, q) processes, in terms of the mathematical structures described in Section A.1, are either:

a) All singularities of $\frac{\Theta(z)}{\Phi(z)}$ outside the unit circle are removable, and

$$\mathbb{E}_{R_k}\left\{\log^+(|R_k|)\right\} := \mathbb{E}_{R_k}\left\{\max\left(0, \log\left(R_k\right)\right)\right\} < +\infty. \tag{59}$$

b) All singularities of $\frac{\Theta(z)}{\Phi(z)}$ in $\mathbb{C}$ are removable.

Note that a complex number $\tilde{z} \in \mathbb{C}$ is a singularity of $\frac{\Theta(z)}{\Phi(z)}$ if it is such that $\Phi(\tilde{z}) = 0$, in which case it will make $\frac{\Theta(\tilde{z})}{\Phi(\tilde{z})} = +\infty$, undefined. If it is also true that $\Theta(\tilde{z}) = 0$, then we will say that *"$\tilde{z}$ is removable"*, since it would be both the root of $\Theta(z)$ and $\Phi(z)$, allowing the cancellation of the corresponding factors in $\frac{\Theta(z)}{\Phi(z)}$.

The bound established in condition a) is easily verifiable when the noise of the ARMA(p, q) process follows a zero-mean Gaussian distribution, that is, $R_k \sim \mathcal{N}(0, {\sigma_r}^2)$. Indeed, if we apply Jensen's inequality,

$$\mathbb{E}_{R_k}\left\{\log^+(|R_k|)\right\} \leq \log^+\left(\mathbb{E}_{R_k}\left\{|R_k|\right\}\right) \tag{60}$$

$$< \log^+\left(\sigma_r \sqrt{\frac{2}{\pi}}\right) < +\infty. \tag{61}$$