Modeling localized bearing faults using inverse Gaussian mixtures

Pavle Boškoski, Đani Juričić

Jožef Stefan Institute, Jamova cesta 39, SI-1000 Ljubljana, Slovenia
pavle.boskoski@ijs.si
dani.juricic@ijs.si

ABSTRACT

Localized bearing faults exhibit specific repetitive vibrational patterns. Due to the constant angular distance between the roller elements, the vibrational patterns occur on regular angular intervals. Under constant operating conditions such patterns become easily detectable as "periodic" events. Slippage or small variation in rotational speed are commonly modeled by introducing normally distributed time variations, which allows for occurrence of "negative" time intervals. In this paper we present an approach which models the occurrences of localized bearing fault patterns as a realization of random point process whose inter-event time intervals are governed by inverse Gaussian mixture. Having support on $(0, \infty)$, the random impact times can acquire strictly positive values. The applicability of the model was evaluated on vibrational signals generated by bearing models with localized surface fault.

1. INTRODUCTION

Bearing faults are one of the most common causes for mechanical failures (MRWG, 1985; Albrecht, Appiarius, & Shrama, 1986). Consequently, the majority of the proposed fault detection methods address the issue of bearing fault detection. Commonly, the well adopted methods focus on extracting and analyzing the behavior of a set of features that describe bearing surface faults, so-called bearing fault frequencies (Tandon & Choudhury, 1999). Inferring about bearing condition using such a feature set is possible if the monitored bearing is operating under constant rotational speed. However, rotational speed fluctuations, which are quite common in real world, reduce the effectiveness of these features. In this paper we model the vibrational patterns generated by bearings with localized surface fault modeling as a point process with inverse Gaussian mixture inter-event distribution.

From a practical point of view, condition monitoring of bearings operating under variable regimes is the most plausible real world scenario. As a result, recently many authors proposed new approaches for condition monitoring of machinery operating under non-stationary regimes (Zhan, Makis, & Jardine, 2006; Combet & Zimroz, 2009; Wang, Makis, & Yang, 2010; Boškoski & Juričić, 2012a; Cocconcelli, Bassi, Secchi, Fantuzzi, & Rubini, 2012; Boškoski & Juričić, 2012b; Heyns, Godsill, Villiers, & Heyns, 2012). Despite the non-stationarity of the generated vibrations, these approaches manage to exploit the statistical properties of some specific vibrational patterns, hence performing sufficiently accurate condition monitoring. Focusing on bearing fault detection, the main source of information are the time occurrences of particular vibrational patterns. Based on the statistical properties of these time occurrences several effective fault detection methods have been developed (Antoni & Randall, 2003; Borghesani, Ricci, Chatterton, & Pennacchi, 2013). In the same manner we propose an approach that describes the impacts generated by localized bearing surface damage as a realization of a point process whose inter-event times are governed by pure or inverse Gaussian mixture.

Initially, inverse Gaussian distribution was developed by Schrödinger (1915) as the distribution of the first passage time of a Wiener process with positive drift and fixed threshold. The first detailed in-depth analysis of the statistical properties of inverse Gaussian distribution was derived much later by Tweedie (1957) and afterwards by Folks and Chhikara (1978). Since then inverse Gaussian distribution has been applied in many different areas for instance: production modeling (Desmond & Chapman, 1993), reliability (Lemeshko, Lemeshko, Akushkina, Nikulin, & Saaidia, 2010), neural spike train modeling (Vreeswijk, 2010), condition monitoring (Boškoski & Juričić, 2011) etc. Although in many cases the application of pure inverse Gaussian model suffices, in this paper we show that under variable rotational speed inverse Gaussian mixture is more suitable model for describing localized bearing faults.

The paper is organized as following. Section 2 contains the definition and the basic statistical properties of the inverse Gaussian distribution. The selection between models describing pure or mixture of inverse Gaussian distributions is presented in Section 3. The actual modeling of localized bearing faults with in the proposed framework is presented in Section 4. Finally, the experimental validation of the models is given in Section 5.

2. Pure and mixed inverse Gaussian distributions

2.1. Pure inverse Gaussian distribution

Let a stochastic process $\alpha(t)$ be

$$\alpha(t) = \nu t + \sigma^2 W(t),\ \nu > 0, \tag{1}$$

where ν is the positive drift, σ^2 is the variance and $W(t)$ is Wiener process (Matthews, Ellsworth, & Reasenberg, 2002). Schrödinger (1915) showed that the first passage time of the process (1) over a fixed threshold a follows the Inverse Gaussian distribution (Folks & Chhikara, 1978):

$$f(t;\mu,\lambda) = \sqrt{\frac{\lambda}{2\pi t^3}}\exp\left(-\frac{\lambda(t-\mu)^2}{2\mu^2 t}\right), \quad t>0, \mu = a/\nu > 0, \lambda = a^2/\sigma^2. \tag{2}$$

Since the parameters μ and λ in (2) are time invariant, the resulting stochastic process is stationary. A simple realization of such a process is shown in Figure 1.

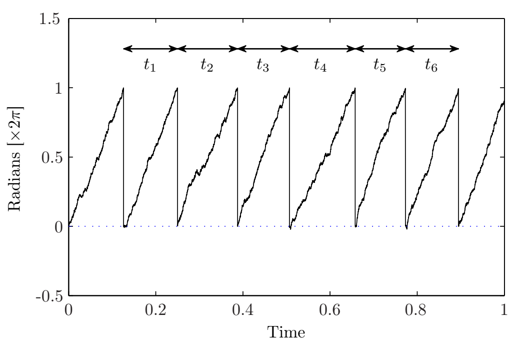

Figure 1. Simulated realization of the stochastic process (1). The time intervals t_i are distributed by inverse Gaussian distribution (2)

2.2. Mixed inverse Gaussian distribution

When modeling data generated by Wiener process (1) there are many situations in which parameters μ and λ in (2) should be considered as random variables. Under such circumstances, the distribution of the first passage time can be described by inverse Gaussian mixtures (Whitmore, 1986). Physically more sound is to allow the positive drift ν in (1) to vary randomly according with some pre-defined distribution. In order to keep the relation with the positive drift ν more clearly visible, Desmond and Chapman (1993) reparametrized (2) by setting $\delta = 1/\mu$:

$$f(t;\delta,\lambda) = \sqrt{\frac{\lambda}{2\pi t^3}}\exp\left(-\frac{\lambda(\delta t-1)^2}{2t}\right), \tag{3}$$

where $t > 0, \delta > 0, \lambda = a^2/\sigma^2$.

In such a form the parameter δ is linearly related to the positive drift ν in (1). By allowing δ to be random variable with distribution $p_\delta(\delta)$, the marginal distribution reads:

$$h(t;\theta) = \int_\triangle f(t;\lambda|\delta)p_\delta(\delta)\,d\delta, \tag{4}$$

where θ is the vector comprising of λ and all hyper parameters of $p_\delta(\delta)$.

3. Model selection

The likelihood functions (2) and (4) specify two different models M_1 and M_2 respectively that can be used for describing the time occurrences t. The selection of which model is more appropriate can be performed by using Bayes' factor.

The application of the Bayes' factor incorporates the concepts of parsimony, unlike the standard likelihood which suffers from the problems of overfitting (MacKay, 2005; Berkes & Fiser, 2011). For the observed data t the Bayes' factor between two models M_1 and M_2 reads:

$$\frac{P(M_1|t)}{P(M_2|t)} = \underbrace{\frac{P(t|M_1)}{P(t|M_2)}}_{Bayes factor} \times \frac{P(M_1)}{P(M_2)}, \tag{5}$$

where $P(M_1)$ and $P(M_2)$ are prior distributions associated with each model.

The two likelihoods entering the Bayes' factor can be calculated by integrating over the complete set of parameters as:

$$\begin{aligned} P(t|M_1) &= \int f(t|\theta_1, M_1)p(\theta_1|M_1)\,d\theta_1 \\ P(t|M_2) &= \int h(t|\theta_2, M_2)p(\theta_2|M_2)\,d\theta_2, \end{aligned} \tag{6}$$

where $f(t|\theta_1)$ is defined by (2), $h(t|\theta_2)$ is defined by (4) and θ_1 and θ_2 are their corresponding parameter sets.

3.1. Specification of the prior $p_\delta(\delta)$

In order to complete the calculation of the Bayes' factor (5), one has to specify the distribution of the random positive drift δ in (4). One possible model of the drift fluctuations, similar to the one specified by Desmond and Yang (2011), reads:

$$\delta = d + \varepsilon,\ \text{where } \varepsilon \sim \mathcal{N}(0,\sigma_\delta^2), d \geq 0, \delta > 0. \tag{7}$$

For cases when the parameter $\sigma_\delta = 0$, the drift parameter δ becomes deterministic, thus the mixture inverse Gaussian (4) reduces into its standard form (2).

The limitation $\delta > 0$ imposes additional limitation on the distribution of ε in (7). Consequently, one has to use Gaussian distribution of ε truncated so that $\varepsilon > -d$.

Using the model (7) with truncated Gaussian distribution as a prior for the speed fluctuations, the marginal likelihood (4) becomes:

$$\hat{h}(t;\lambda,\sigma_\delta,d) = \sqrt{\frac{\lambda}{2\pi t^3(1+\lambda\sigma_\delta^2 t)}} \times \exp\left(-\frac{\lambda(dt-1)^2}{2t(1+\lambda\sigma_\delta^2 t)}\right) \times \frac{\Phi\left(\frac{d+\lambda\sigma_\delta^2}{|\sigma_\delta|\sqrt{1+\lambda\sigma_\delta^2 t}}\right)}{\Phi\left(\frac{d}{\sigma_\delta}\right)}, \tag{8}$$

where $\Phi(\cdot)$ is the cumulative function of the standard normal distribution.

The proposed speed model (7) defines random and stationary speed profile. When necessary, an arbitrary speed profile can be used instead. The only problem would be to specify a proper definition of the prior $p_\delta(\delta)$ and calculate new marginal likelihood (8).

Finally, it has to be emphasized that the modeled parameter in (7) is the standard deviation σ_δ instead of the variance. By modeling through the variance an additional limitation will be imposed i.e. $\sigma_\delta^2 \geq 0$. Such a parametrization introduces a limitation since the parameter under null hypothesis $\sigma_\delta^2 = 0$ lies on the limit of the acceptable region. Therefore standard likelihood tests become inapplicable (Lehmann & Casella, 1998, Chapter 5).

4. Bearing fault detection by means of inverse Gaussian models

Bearing faults are surface damages that occur on the bearing elements. Each time when a rolling element passes over the damaged surface, a specific vibrational pattern is generated directly connected to one of the bearings eigenmodes. Usually, under constant operating conditions the generated vibrations are modeled as (Randall, Antoni, & Chobsaard, 2001):

$$x(t) = \sum_i A_i s(t - iT - \tau_i), \tag{9}$$

where A_i is the amplitude of the i^{th} impact, $s(t)$ is the impulse response of the excited eigenmode, T is the period of rotation and τ_i is random fluctuation due to slippage. Generally, τ_i is modeled as zero mean normally distributed with sufficiently small variance σ_τ^2. Regardless of the variance σ_τ^2, model (9) allows for τ_i to acquire sufficiently low negative values. Consequently, the occurrence of the $i+1^{\text{th}}$ impact might be modeled as if it occurs before the i^{th} one.

4.1. Using inverse Gaussian distribution

Avoiding the issues of negative time delays, present in model (9), we propose the following model of generated vibrations:

$$x(t) = \sum_i A_i s(t - t_i), \tag{10}$$

where A_i is the amplitude of the ith impact, $s(t)$ is the impulse response of the excited eigenmode and t_i is the time of the occurrence modeled as inverse Gaussian random variable. A typical vibrational pattern is shown in Figure 2.

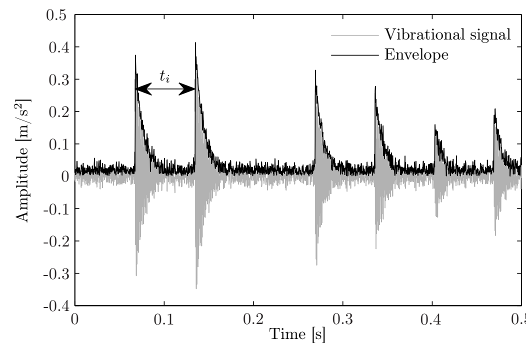

Figure 2. Simulated conceptual vibrational pattern generated by damaged bearing

Due to the mechanical characteristics of the bearings, the angular distance between the adjacent rolling elements is constant. Therefore, the angular distance between two consecutive impacts can be regarded as constant too. So, one can easily apply the stochastic process (1) to model the angular distance traveled by a rolling element towards the damaged surface. The threshold a in (1) is the actual angular distance between the roller elements and ν is directly related to the rotational speed. Consequently, the time intervals t_i between two adjacent excitations of $s(t)$ can be modeled as a realization of either pure or mixture inverse Gaussian, depending on the statistical characteristics of the rotational speed.

Pure inverse Gaussian model (2) for the inter-impact times t_i should be regarded as a special case, valid when the bearing rotational speed is "constant" i.e. there are no significant speed fluctuations. Under such circumstances pure inverse Gaussian model (2) is applicable for localized bearing surface faults (Boškoski & Juričić, 2011).

A more realistic scenario is the one where the rotational speed of a bearing varies randomly. Under such circumstances the angle covered by a rolling element can be modeled as a realization of the stochastic process (1) by allowing the positive drift $\nu \propto \delta$ to vary randomly according to the random speed fluctuations. Consequently, the observed time intervals t_i between two consecutive impacts can be modeled as a realization of an inverse Gaussian mixture (8).

4.2. Multiple localized faults

The case of multiple localized surface faults can be also described in the framework of point processes with inverse Gaussian inter-event distribution. For that purpose one can consider a Wiener process, similar to (1), with two barriers a and b. Starting from an initial point the time required to reach the barrier a is T_1, and time to reach the barrier b from a is T_2. Chhikara and Folks (1989) showed that T_1 and T_2 are independent inverse Gaussian random variables defined as:

$$\begin{aligned} T_1 &\sim IG\left(\frac{a}{\nu}, \frac{a^2}{\sigma^2}\right) \\ T_2 &\sim IG\left(\frac{b-a}{\nu}, \frac{(b-a)^2}{\sigma^2}\right) \end{aligned} \tag{11}$$

Measuring from the initial starting point reaching the threshold b can be specified as time $T_3 = T_1 + T_2$. Since the ratio

$$\frac{\lambda_i}{\mu_i^2} = \frac{\nu^2}{\sigma^2} = \text{const.}, \tag{12}$$

the time T_3 is also inverse Gaussian random variable distributed as (Chhikara & Folks, 1989):

$$T_3 \sim IG\left(\mu_1 + \mu_2, \frac{\nu^2(\mu_1+\mu_2)^2}{\sigma^2}\right). \tag{13}$$

In the context of bearings, the threshold a is the angular distance of the first fault in the direction of rotation measured from some initial point. The threshold b, on the other hand, is the angular distance measured from the first fault in the direction of rotation.

By extending the concept of two thresholds (13) to multiple thresholds, one can model multiple localized bearing faults by employing the generalized distribution of inter-event times (Chhikara & Folks, 1989, Chapter 11).

5. EXPERIMENTS

The proposed model based on mixture of inverse Gaussian distribution of the inter-event times was evaluated on simulated vibration signals. The signals were generated using the dynamic bearing model developed by Sawalhi, Randall, and Endo (2007) enhanced with the EHL (Elastohydrodynamic Lubrication) model developed by Sopanen and Mikkola (2003a, 2003b). The simulated bearing had localized surface fault on the outer ring. The fault was simulated to be 2° wide and has average surface depth of $30\mu m$.

Simulations were performed using several different speed profiles according to the model (7) with mean value d = 38 Hz. The standard deviation σ_δ changed from 0% up to 10% of the mean speed d.

5.1. Detection of impacts times

The main information required for the application of proposed inverse Gaussian based models are the time intervals between two consecutive impacts. Therefore, the first step in the analysis is the detection of impact times. In our approach, the detection of impact times was performed using wavelet transform thresholding. The main parameter that has to be selected is the mother wavelet. Schukin, Zamaraev, and Schukin (2004) suggested that for signals containing repetitive impulse responses, an optimal detection of impacts can be performed by using mother wavelet that will closely match the underlying vibrational patterns. However, Unser and Tafti (2010) provided thorough analysis that the crucial parameter for sparse wavelet representation of signals containing repetitive impulse responses, is the number of vanishing moments of the mother wavelet rather then the selection of the "optimal" mother wavelet that will closely match the underlying signal. Therefore, by selecting a wavelet with sufficiently high number of vanishing moments one can sufficiently accurately analyze vibrational patterns containing the impulse responses from the excited eignemodes regardless of their variable form due to the changes of the transmission path. The schematic representation of the impact detection process is shown in Figure 3.

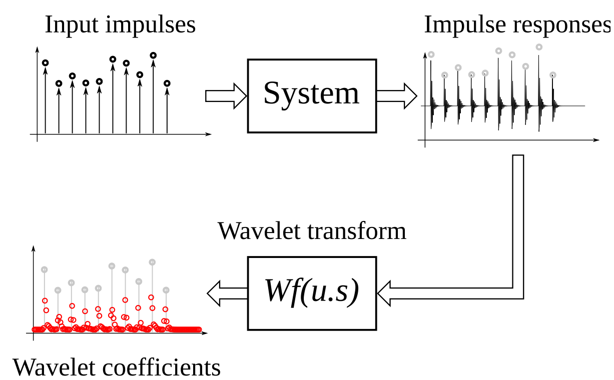

Figure 3. Detection of impact times using wavelet as differential operator

In our approach, the generated vibrations were analyzed using Daubechies 10 mother wavelet (Daubechies, 1992). For our particular system such a number of vanishing moments has shown to be sufficient for accurate impulse detection.

5.2. Numerical calculation of the Bayes' factor

Having the impact times t_i the next step is to calculate the Bayes' factor by calculating the marginal distributions (6). The marginal likelihoods were calculated using Monte Carlo integration. Since the model selection depends on the standard deviation σ_δ (not the variance σ_δ^2), the selected prior was so-called folded non-central t distribution (Gelman, 2006) which reads:

$$p(\sigma_\delta) \propto \left(1 + \frac{1}{\gamma}\left(\frac{\sigma_\delta}{A}\right)^2\right)^{-(\gamma+1)/2}, \qquad (14)$$

where A is scale parameter and γ represents the degrees of freedom. The prior for the mean value d in (7) was chosen to be uniform in sufficiently wide interval. The prior for the remaining parameter $\lambda = 1/\sigma^2$ in (1) was also chosen to be uniform in the interval that contains 2% of initial speed fluctuations due to slippage (Randall et al., 2001).

5.3. Experimental results

One realization of the speed fluctuations, modeled according to (7) with $d = 38$ Hz, is shown in Figure 4. The speed fluctuations are smooth but sufficiently fast. Consequently even during a single bearing revolution the rotational speed varies.

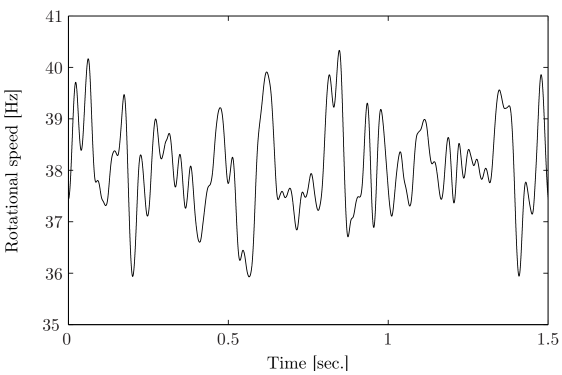

Figure 4. A typical speed fluctuation profile

For small speed deviations $\sigma_\delta < 0.5\%$ of the mean speed value d, the Bayes' factors (6) overwhelmingly favor simpler model (2) i.e. pure inverse Gaussian distribution of the inter-event times. For speed fluctuations with $\sigma_\delta > 0.5\%$ the Bayes' factors favor mixture inverse Gaussian model for the inter-event times. Changes of the Bayes' factor with respect to the changes in the speed fluctuations σ_δ are shown in Figure 5.

Such results are somewhat expected since under small speed fluctuations pure inverse Gaussian distribution of the inter-event times sufficiently well describes the observed impact times. At the same time, due to the principle of parsimony, the simpler model is preferred. The cost of more complex model becomes justified when the speed fluctuations become more intense.

5.4. Comments on results

The effectiveness of the proposed approach becomes apparent if one compares it with other methods. Due to the random speed fluctuations, the standard spectral methods are inapplicable and the only choice is time-frequency analysis of the signal. Therefore, we calculated the wavelet transform of the envelope of the generated vibrations, which is shown in Figure 6. One can easily notice that the envelope contains some patterns in the vicinity of 90 Hz. However, the patterns exhibit no particular structure and it is quite difficult to draw any conclusions from such a time-frequency plot.

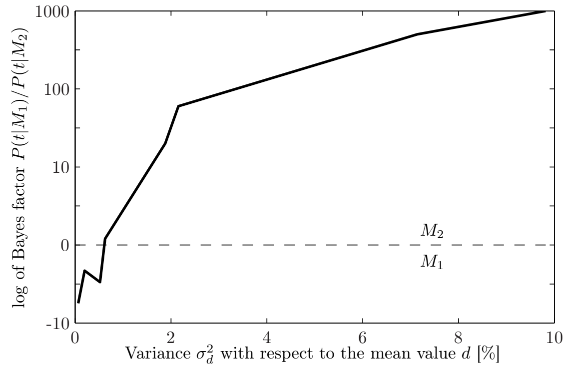

Figure 5. Changes of the Bayes' factors for different σ_δ

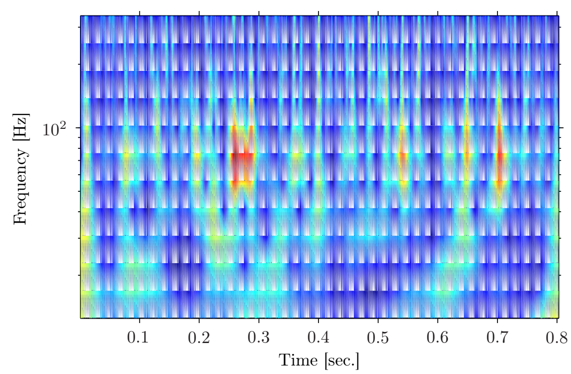

Figure 6. Wavelet transform of the envelope of the generated vibrations

The analysis of the impacts as a realization of a point process with pure or inverse Gaussian mixture offers a framework for proper statistical testing about the origin of the observed events. Testing whether the observed impacts are related to a specific angular position is fairly straightforward. Furthermore, the same analysis offers an insight about the possible mixing distribution, i.e. the distribution of the variable rotational speed.

6. Conclusion

The experimental results show that the specific vibrational patterns generated by bearings with surface faults can be treated as a realization of a point process whose inter-event times are distributed according to either pure or inverse Gaussian mixture. The pure inverse Gaussian distribution is applicable for the special case when fault bearings operate under constant rotational speed. The inverse Gaussian mixture, on the other hand, is a general solution applicable also for modeling the inter-impact times of faulty bearings operating under variable rotational speed. Finally, unlike the commonly adopted models for bearing vibrations, the proposed model is inline with the physical limitations by modeling random time fluctuations with distribution with support on interval $(0, \infty)$.

The application of the proposed approach on acquired vibrations starts by calculating the time intervals between adjacent impacts through the wavelet coefficients calculated from the generated vibration signals. When the observed impacts are generated by a phenomenon that occurs on regular angular intervals, the corresponding inverse Gaussian model can be employed. Determining the validity of such a claim can be performed by a straightforward calculation of the Bayes' factors. This approach is applicable to both constant and variable operating conditions.

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