Equivalent Accelerated Degradation Test Plans in a Nonlinear Random Coefficients Models

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ABSTRACT

Design of optimal Accelerated Degradation Testing (ADT) plan has been extensively researched over several decades. In practice, due to the rapidly changing development and assessment environment, pre-established plans often fail to meet reality. Therefore, designing a test plan that is equivalent to the target plan using an different stress-loading or a testing condition is needed to allow for more flexibility. However, there exists currently little work in the development of equivalent ADT plan. In this paper, we proposes an equivalent cost-effective accelerated degradation test (ADT) plan in the context of a nonlinear random-coefficients model. The proposed model is applied to a well-known constant-stress ADT problem in the literature.

1. INTRODUCTION

Intense global competition in the high-technology industry forces manufacturers to evaluate product reliability within shorter testing times and with limited resources. The most common approach for the purpose is to use an accelerated life test to hasten product failures during test intervals by stressing the product beyond its normal use condition. Recently, (accelerated) degradation tests have replaced traditional (accelerated) life tests. Degradation data not only lead to improved reliability analysis compared to standard failure time analysis (Lu, Meeker, & Escobar, 1996), but they also provide additional information related to failure mechanisms for testing units.

Design of optimal (accelerated) degradation test plan has been extensively researched over several decades and many different types of optimization problems have been presented. However, the experimenter who conducts the experiment can often face the situation that the optimal plan is not feasible due to limited resources and rapidly changing environments (e.g., testing equipments, samples, time, etc.), even after taking into account the design factors in the planning phase of the experiment. Therefore, designing a test plan that is equivalent to the target plan using an different stress-loading or a testing condition is needed. The equivalent plan can not only make the experiment feasible but also provide a more flexible and economic alternative.

The concept of equivalent plan was first described by (Escobar & Meeker, 1995). The authors proposed methods for planning two-factor ALT plans by splitting an optimal degenerate plan while maintaining the same optimality criterion. The concept of equivalent plan, which was utilized in their research, was implicitly used in the special case of ALT plan. Recently, (Elsayed, Zhu, Zhang, & Liao, 2009) proposed an approach to determine the simple-step-stress ALT plan that minimizes the termination time and meets equivalence to a three-level constant-stress ALT plan. (Liao & Elsayed, 2010) proposed a general idea for obtaining the equivalent ALT plans involving different stress loadings and investigated several types of quivalent ALT plans.

Unfortunately, to the best of our knowledge, there exists currently little work in the development of equivalent ADT plan. In this paper, we define the equivalence of ADT plans and propose a cost-effective equivalent ADT plan in the context of a nonlinear random-coeffcients model when the target plan is specified.

2. ADT MODELS

2.1. The ADT Model

We assume that an ADT is conducted under the following conditions:

1. Constant-stress loading is adopted at n stress levels, $x_D = x_L \leq x_1 \leq \cdots \leq x_n \leq x_H$, where x_L and

x_H are the minimum and maximum stress level, respectively. x_D is the use condition level.

2. The total number of testing units is M and m_i units are allocated to each stress level such that $m_i = \pi_i M$ and $\sum_{i=1}^{n} \pi_i = 1$, $\pi_i \geq 0$.
3. The all testing units at a given stress level x_i are measured at the same time. The measurement times, $\boldsymbol{\xi}_i = (t_{i1}, \ldots, t_{il_i})$, are derived by the pre-specified plan that is a function of the number of measurements l_i and the termination time t_{il_i}.

Let $\boldsymbol{\Xi}$ be an ADT plan. It specifies a measurement times, $\boldsymbol{\xi}_i$, the corresponding allocation proportion of testing units, π_i, at stress level x_i and the total number of testing units, M. A general expression of the ADT plan with n stress levels can be denoted by

$$\boldsymbol{\Xi} = \{[\boldsymbol{\xi}_1, x_i, \pi_1], \ldots, [\boldsymbol{\xi}_n, x_n, \pi_n], M\}$$

where $\sum_{i=1}^{n} \pi_i = 1$ and ξ_i is a function of the number of measurements l_i and the termination time t_{i,l_i} at each stress level. The random-coefficients model provides a flexible and powerful tool for analyzing repeated-measurement data and allows for various variance-covariance structures of the response vector.

$$\mathbf{y}_{ij} = \boldsymbol{\eta}(\boldsymbol{\xi}_i; x_i, \boldsymbol{\theta}_{ij}) + \boldsymbol{\epsilon}_{ij}(\boldsymbol{\xi}_i), \quad i = 1, \ldots, n, \; j = 1, \ldots, m_i \tag{1}$$

where $\mathbf{y}_{ij}$ is the $(l_i \times 1)$ vector of observed degradation path of the jth individual at ith the stress level; $\boldsymbol{\eta}(\cdot)$ is the $(l_i \times 1)$ vector of mean responses of the testing unit; $\boldsymbol{\xi}_i$ is the $(l_i \times 1)$ vector of measurement times for the ith stress level; $\boldsymbol{\theta}_{ij}$ is the $(p \times 1)$ vector of individual-specific model parameter; x_i is the ith stress level; $\boldsymbol{\epsilon}_{ij}$ is i.i.d. normally distributed measurement error. We assumed that $\boldsymbol{\eta}(\cdot)$ is a continuous, monotone and nonlinear function of t in $\boldsymbol{\theta}_{ij}$.

2.2. Failure-time Distribution

2.2.1. Asymptotic Variance of Quantile

After pre-specifying the critical level η_c, the failure-time T is defined as the time at which the mean degradation path $\eta(t, \boldsymbol{\theta})$ reaches the critical threshold η_c (i.e., $\eta(t, \boldsymbol{\theta}) \leq \eta_c$). The failure-time distribution for the MDDP at a threshold level η_c is

$$F_T(t|\eta_c) = \Phi\left(\frac{\eta_c - \mu(t; \boldsymbol{\theta})}{\sqrt{V(t; \boldsymbol{\theta})}}\right) \tag{2}$$

where $\mu(t; \boldsymbol{\theta})$ and $V(t; \boldsymbol{\theta})$ are the mean and variance of the degradation data at a specific time t, respectively. Here, $\Phi(\cdot)$ is the cumulative function of a standard normal distribution.

The precision of the pth quantile (denoted by t_p) of the failure-time distribution is the main concern of reliability analysis. From (2), t_p for the MDDP is obtained by solving $\eta_c = \mu(t; \boldsymbol{\theta}) + \Phi^{-1}(p)\sqrt{V(t; \boldsymbol{\theta})} \equiv h(t; p, \boldsymbol{\theta})$.

The asymptotic variance of the pth quantile for the failure-time distribution is

$$\text{AVar}_p(\boldsymbol{\Xi}) = \frac{\partial \mathbf{G}(\boldsymbol{\theta})^T}{\partial \boldsymbol{\theta}} \mathbf{I}^{-1}(\boldsymbol{\Xi}|\boldsymbol{\Psi}) \frac{\partial \mathbf{G}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}, \tag{3}$$

where $\mathbf{I}(\boldsymbol{\Xi}|\boldsymbol{\Psi})$ is the $(2p \times 2p)$ upper-left population Fisher information matrix and $\mathbf{G}(\boldsymbol{\theta}) = h^{-1}(\eta_c; p, \boldsymbol{\theta})$ is the precision of t_p.

3. EQUIVALENT ADT PLANS

3.1. Definition of Equivalence

We can say that two ADT plans are equivalent if they have same criteria under the condition that they satisfy all the constraints and other criteria. The precision of the pth quantile of the failure-time distribution at use condition is the most used in practical testing plans.

[Definition] (Equivalent ADT Plan) Two ADT plans are equivalent if the relative error of the precision of the reliability prediction values (e.g., AVar) or the optimization criterion is less than $\delta(\delta > 0)$ under the condition that they satisfy all the constraints and other auxiliary equivalent criteria.

Let $\mathbf{d}_E$ be the decision variable vector of the equivalent ADT plan.Let $\mathbf{I}_T$ and $\mathbf{I}_E$ be the Fisher information matrices of the target ADT plan and the equivalent ADT plan, respectively. If we set the precision of the pth quantile of the failure-time distribution as equivalent measure, the equivalent ADT will satisfy following conditions:

$$\frac{\left|\Delta\mathbf{G}^T\mathbf{I}_E^{-1}\Delta\mathbf{G} - \Delta\mathbf{G}^T\mathbf{I}_T^{-1}\Delta\mathbf{G}\right|}{\Delta\mathbf{G}^T\mathbf{I}_T^{-1}\Delta\mathbf{G}} < \epsilon, \qquad \text{where } \epsilon \geq 0 \tag{4}$$

$$K_{q,E} = K_{q,T}, \qquad q = 1, 2, \ldots$$

where $\Delta\mathbf{G} = \partial\mathbf{G}(\boldsymbol{\theta})^T / \partial\boldsymbol{\theta}$ is a partial derivative of $\Delta\mathbf{G}$ with respect to paratemeters $\boldsymbol{\theta}$ and $K_{q,E} = K_{q,T}$ is auxiliary equivalent criteria, such as total number of testing units, experimental cost, and termination time.

3.2. Formulation of Equivalent ADT Plan

3.2.1. Cost Function

The experimental cost can be defined as

$$\text{COST}(\boldsymbol{\Xi}) = M \times \left[C_s + \sum_{i=1}^{n} \pi_i \left\{C_T(t_{i,l_i}) + l_i \times C_{\mathrm{I}}\right\}\right], \tag{5}$$

where l_i is the number of measurements, C_{I} is a measurement cost, C_s is a sample cost, t_{i,l_i} is a termination time for i-th stress level, and C_T is an operating cost function depending on the termination time of degradation test at each stress levels. The operating cost function is defined as

$$C_T(t) = \begin{cases} c_1 + c_2 \times t & 0 \le t \le t_a \\ [c_1 + c_2 \times t_a] \times e^{c_3(t-t_a)} & t > t_a, \end{cases} \tag{6}$$

where c_1 is a fixed cost, c_2 is an operating cost per unit time and c_3 is a delay-penalty coefficient. Generally, if time-to-market (denoted by t_a) for new products cannot be satisfied, the opportunity cost will increase nonlinearly. We introduce the cost function (5) to reflect this real situation; that is, if the experiment continues after the critical time t_a, an additional delay cost is nonlinearly imposed on the operational cost function.

3.2.2. Optimization Problem

The equivalent ADT plan can exist in a variety of forms. In this paper, we consider a 3-level cost-effective equivalent ADT plan based on the definition. The goal of the optimization formulation is to identify the best combination of decision variable values that meets equivalent criteria and testing constraints at minimum cost.

We assume that a target testing plan is given by a preliminary study. Suppose that $x_3 = x_H$, $x_2 = (x_1 + x_3)/2$, and $\pi_1 = 4/7$, $\pi_2 = 2/7$, $\pi_3 = 1/7$, which is commonly used proportion ratio proposed by (Meeker & Hahn, 1985), are pre-specified. As an auxiliary criterion for equivalence, we consider the equivalent ADT plan has the same number of total testing unit as the target plan, i.e. $M = M_T$. The decision variables become $\mathbf{d}_E = (t_{1,l_1},\ t_{2,l_2},\ t_{3,l_3},\ l_1,\ l_2\ , l_3,\ x_1)$. Then, the optimization problem for 3-level equivalent ADT plan can be formulated as,

$$\begin{aligned} \underset{\mathbf{d}_E}{\text{minimize}} \quad & \mathrm{COST}(\mathbf{d}_E) \\ \text{subject to} \quad & \frac{\left|\Delta\mathbf{G}^T\mathbf{I}_E^{-1}\Delta\mathbf{G} - \Delta\mathbf{G}^T\mathbf{I}_T^{-1}\Delta\mathbf{G}\right|}{\Delta\mathbf{G}^T\mathbf{I}_T^{-1}\Delta\mathbf{G}} < \delta,\ \delta \ge 0 \\ & M = M_T, \\ & x_L < x_1 < x_3 = x_H,\ \ x_2 = (x_1 + x_3)/2, \\ & l_{min} \le l_i,\ \ t_{min} \le t_{i,l_i} \le t_{max},\ i = 1,2,3. \end{aligned} \tag{7}$$

The measurement times, $\xi_i = (t_{i,1}, \cdots, t_{i,l_i})$, $i = 1,2,3$, are derived by one of the heuristic plans: ED(Equal Degradation), EL(Equal Log-spacing), and ES(Equal Spacing).

4. NUMERICAL EXAMPLE

4.1. Device-B Example

In this section, we will illustrate application of the approahces proposed in the preceding section to integrated circuit(IC) devices called "Device-B" in (Meeker & Escobar, 1998). The purpose of the experiment is to estimate the 10th quantile of failure-time distribution under the use temperature ($80\,^\circ$C). The highest operating temperature of the device is $237\,^\circ$C. Failure of the device is defined as power output drops below -0.5 decibels(dB). In the target testing plan, devices were allocated at each of three temperatures: $(150\,^\circ\mathrm{C}, 7\text{ units})$, $(195\,^\circ\mathrm{C}, 12\text{ units})$, $(237\,^\circ\mathrm{C}, 15\text{ units})$, respectively. The termination times are 4000, 2000, and 1000 hours, respectively. As ES(Equal Spacing) plan and same measurement frequency (i.e., 125 hours) are adopted in the all temperature levels, the number of measurements are 33, 17, 9, respectively. Figure 1 shows the ADT data of power output at each temperature. Please refer to (Meeker & Escobar, 1998) for details of data and model.

4.2. The Cost-Effective equivalent ADT plan for Device-B

Let $\boldsymbol{G}(\boldsymbol{\theta})$ be the 10th quantile of the failrue-time distribution at use condition. To obtain the equivalent ADT plan for minimum cost function, we set the lower and upper bound of the measurement time as $t_{min} = 170$ and $t_{max} = 4,000$ hours, respectively. The minimum number of measurement times, l_{min}, is set as 3. The permissible relative error between the asymptotic variances is less than 0.001, i.e. $\delta = 0.001 = 0.1\%$. The coefficients of cost function are set as: $(C_s, C_I, c_1, c_2, c_3, t_1) = (50, 10, 0, 0.3360, 9.316 \times 10^{-4}, 2976)$. Then, the optimization problem for 3-level equivalent ADT plan is expressed as

$$\begin{aligned} \underset{\mathbf{d}_E}{\text{minimize}} \quad & \mathrm{COST}(\mathbf{d}_E) \\ \text{subject to} \quad & \frac{\left|\Delta\mathbf{G}^T\mathbf{I}_E^{-1}\Delta\mathbf{G} - \Delta\mathbf{G}^T\mathbf{I}_T^{-1}\Delta\mathbf{G}\right|}{\Delta\mathbf{G}^T\mathbf{I}_T^{-1}\Delta\mathbf{G}} < 0.001 \\ & M = 34, \\ & 80 < x_1 < x_3 = 237,\ \ x_2 = (x_1 + x_3)/2, \\ & 3 \le l_i,\ \ 170 \le t_{i,l_i} \le 4000,\ i = 1,2,3 \end{aligned}$$

where $\mathbf{I}_E$ and $\mathbf{I}_T$ are the Fisher information matrix of the equivalent plan and the target plan, respectively.

The cost-effective equivalent ADT plans are obtained in Table 1. From the result, we can see that the equivalent plans significantly reduce both the termination time and experimental cost. Reductions of the time and cost relative to the target plan with respect to the type of measurement plans(ED, EL, ES) are (91.05%, 80.57%), (87.96%, 76.69%), and (88.99%, 79.12%), repectively. In this case, the test plan based on the ED measurement strategy is the most cost-effective and time-saving equivalent plan.

5. CONCLUSION

In this paper, we propose a cost-effective equivalent ADT plan in the context of a nonlinear random-coefficients model. The definition of equivalence is proposed to design equivalent ADT plan. The cost function is adopted which balances

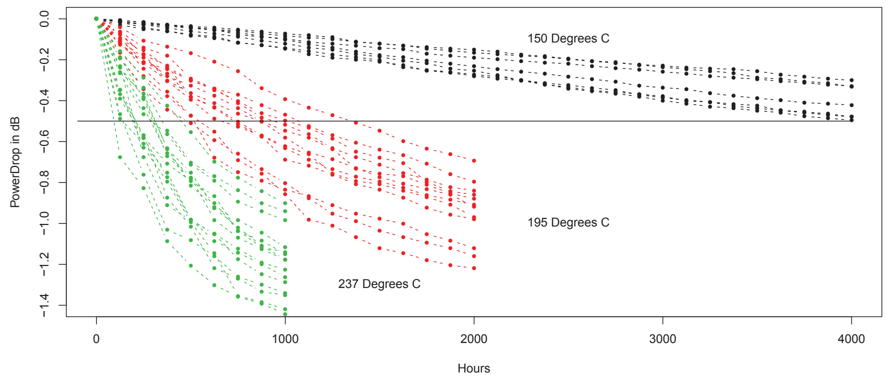

Figure 1. ADT data of power drop in Device-B.

Table 1. Cost-effective equivalent ADT plans.

Plan parameters	Target plan	Equivalent plan		
		ED	EL	ES
Cost	35,605	6,917	8,300	7,434
δ	-	1.27×10^{-5}	1.84×10^{-5}	1.33×10^{-5}
t_{1,l_1}	4000	358.18	481.69	303.86
t_{2,l_2}	2000	170.00	170.00	170.00
t_{3,l_3}	1000	292.66	348.84	440.27
x_1	150	144.66	144.38	144.85
(l_1, l_2, l_3)	$(33, 17, 9)$	$(6, 4, 6)$	$(9, 3, 6)$	$(8, 7, 5)$
(π_1, π_2, π_3)	(7/34,12/34,15/34)		(4/7,2/7,1/7)	
M	34		34	
Cost reduction	-	80.57%	76.69%	79.12%
Time reduction	-	91.05%	87.96%	88.99%

between time-to-market and experimental resources. In the well-known constant-stress ADT example in the literature, the proposed equivalent plan dramatically reduces both the total cost and the termination time while maintating the same precision of the pth quantile of the failure-time distribution at use condition and the number of total testing units. Although this work considers a specific example and a general definition, the equivalent ADT design approach can be applied through a wide range of types of practical applications. An equivalent ADT plan under multi-stress, different stress-loading, or different equivalence definition can be one of the possible future researches.

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