# Comparing the parameter estimation methods of Weibull distribution with censored lifetime

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## ABSTRACT

Weibull distribution is widely used in reliability engineering and lifetime analysis because of its flexibility in modeling both increasing and decreasing failure rate. Weibull distribution has shape parameter and scale parameter, and it is difficult to estimate the parameters due to the no-closed form of likelihood function. In recent years, there has been studied on the approximating parameter estimation methods based on the simulation. In this study, we use the approximating parameter estimation of Weibull distribution with censored lifetime. The methods which are applied in numerical example are Bayesian estimation method, maximum likelihood estimation, and Markov chain Monte Carlo. Accuracy of estimation methods is performed by the mean square errors of parameter estimator in simulation reducing the experiment time. In addition, it can be helpful to set the design of experiment considering the characteristics of Weibull distribution with censored lifetime.

## 1. INTRODUCTION

Parameter estimation is an essential procedure in engineering problems for safety and reliability analysis. In order to improve the reliability of systems, a sufficient amount of sample data will be collected and analyzed (An & Choi, 2013; Tan, 2009). However, there are many limitations on the experimental environment and costs for analysis. Moreover, lifetime data are often censored. Under such constraints, it is necessary to identify the failure mode and predict the residual lifetime and failure by using appropriate estimation methods.

Time to failure is commonly regarded as a random variable following a specific distribution (Jia, Wang, Jiang, & Guo, 2016). In this study, we assume that the lifetime of units follows a Weibull distribution, which is widely used in problems of reliability engineering. Many methods have been proposed to estimate the parameters of Weibull distribution. The most frequently used methods are maximum likelihood estimation and least-squares estimation. In recent years, there has been studied on approximating parameter estimation methods based on the simulation with the development of computers such as Markov chain Monte Carlo (Kelly & Smith, 2011; Ntzoufras, 2011).

This study is to compare the parameter estimation methods of Weibull distribution with censored lifetime. We propose an appropriate parameter estimation method by applying maximum likelihood estimation (MLE), Bayesian estimation (BA), which is traditional parameter estimation method, and Markov chain Monte Carlo (MCMC) method in the numerical examples following Weibull distribution. MCMC method is a powerful simulation technique for exploring high-dimensional probability distributions.

This paper is organized as follows. Section 2 presents Weibull lifetime parameter estimation methods such as maximum likelihood estimation, Bayesian estimation, and Markov chain Monte Carlo methods. In Section 3, numerical study is conducted to compare the parameter estimation methods. Section 4 shows the conclusion.

## 2. WEIBULL LIFETIME PARAMETER ESTIMATION

In this paper, the probability density function of a Weibull random variable, $t$, is defined as

$$f(t;\theta,\alpha)=\begin{cases}\dfrac{\alpha}{\theta}\left(\dfrac{t}{\theta}\right)^{\alpha-1}e^{-\left(\frac{t}{\theta}\right)^{\alpha}}, & t\ge 0\\ 0, & t<0\end{cases} \quad (1)$$

where $\alpha > 0$ is the shape parameter and $\theta > 0$ is the scale parameter of the distribution.

Lifetime of units is often censored (Rinne, 2008). Suppose $n$ units are one or more sets of right censored. A set of data is formalized as $n$ units of independent identically distributed random variable $(\mathrm{Y}_i, \delta_i),\ i = 1, 2, \ldots, n,$ where

$$Y_i = \begin{cases} \mathrm{T}_i, & \delta_i = 1,\ (uncensored) \\ \min(T_i, C_r), & \delta_i = 0,\ (censored)\ , \end{cases} \tag{2}$$

$r$ units fail and $n - r$ units do not fail within a time limit, $C_r$.

### 2.1. Maximum Likelihood Estimation (MLE)

Maximum likelihood estimation (MLE) is a method of estimating unknown parameters of probability distributions which are obtained by maximizing the likelihood function. The likelihood function of Weibull distribution is

$$\begin{aligned} L(\theta, \alpha) &= \prod_{i=1}^{n} f(t_i; \theta, \alpha) \\ &= \left(\frac{\alpha}{\theta^\alpha}\right)^n \left(\prod_{i=1}^{n} t_i^{\alpha-1}\right) \exp\left(-\frac{1}{\theta^\alpha} \sum_{i=1}^{n} t_i^{\alpha}\right). \end{aligned} \tag{3}$$

We apply the natural logarithm of likelihood function, it is

$$\begin{aligned} \ln L(\theta, \alpha) &= l(\theta, \alpha) \\ &= n \ln \alpha - \alpha n \ln \theta + (\alpha - 1) \sum_{i=1}^{n} \ln t_i - \frac{1}{\theta^\alpha} \sum_{i=1}^{n} t_i^\alpha. \end{aligned} \tag{4}$$

By taking the partial derivatives with respect to $\theta$ and $\alpha$, we have the following two equaions,

$$\frac{\partial l(\theta, \alpha)}{\partial \theta} = -\frac{\alpha n}{\theta} + \frac{\alpha}{\theta^{\alpha+1}} \sum_{i=1}^{n} t_i^\alpha, \tag{5}$$

$$\frac{\partial l(\theta, \alpha)}{\partial \alpha} = \frac{n}{\alpha} - n \ln \theta + \sum_{i=1}^{n} \ln t_i - \sum_{i=1}^{n} \left(\frac{t_i}{\theta}\right)^\alpha \ln\left(\frac{t_i}{\theta}\right). \tag{6}$$

Maximum likelihood estimators of Weibull distribution do not exist in closed form (Pradhan & Kundu, 2014). We have to be obtained by solving two non-linear equations for estimating the unknown parameters. Therefore, the Newton-Raphson algorithm for solving simultaneous equation is used as iterative likelihood techniques (Wong, 1977). Given appropriate initial values $\alpha_{(0)}$ and $\theta_{(0)}$ , the $r$ -th iterative solution is

$$\begin{bmatrix} \alpha_{(r)} \\ \theta_{(r)} \end{bmatrix} = \begin{bmatrix} \alpha_{(r-1)} \\ \theta_{(r-1)} \end{bmatrix} - \begin{bmatrix} \dfrac{\partial^2 \ln L}{\partial \alpha_{(\mathrm{r}-1)}{}^2} & \dfrac{\partial^2 \ln L}{\partial \alpha_{(r-1)} \partial \theta_{(r-1)}} \\ \dfrac{\partial^2 \ln L}{\partial \theta_{(r-1)} \partial \alpha_{(r-1)}} & \dfrac{\partial^2 \ln L}{\partial \theta_{(\mathrm{r}-1)}{}^2} \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{\partial \ln L}{\partial \alpha_{(\mathrm{r}-1)}} \\ \dfrac{\partial \ln L}{\partial \theta_{(\mathrm{r}-1)}} \end{bmatrix} \tag{7}$$

where correction term is multiplication of inverse matrix of Hessian (second order derivative) and gradient vector (first order derivative).

### 2.2. Bayesian Estimation (BE)

In the Bayesian approach to estimate the parameters, a natural conjugate prior distribution has been generally used (Kundu & Mitra, 2016). Weibull distribution does not have a conjugate family of join prior distribution due to the no-closed forms (Erto & Giorgio, 2013). However, Weibull distribution with fixed shape parameter $\alpha$ can have the natural conjugate prior distribution. In order to elicit the prior distributions, this study will use re-parameterization of likelihood function, $\theta^\alpha = \lambda$ . The natural conjugate prior distribution is the inverse gamma distribution, which is of the form,

$$\mathrm{p}(\lambda | \nu, \mu) = \frac{\mu^\nu}{\Gamma(\nu)} \lambda^{-\nu-1} e^{-(\mu/\lambda)}, \tag{8}$$

where $\nu$ and $\mu$ are the hyper-parameters in $\lambda > 0$ .

The posterior distribution can be derived by Bayes' formula, given as

$$\begin{aligned} P(\lambda \mid \mathrm{t}_1, \ldots, \mathrm{t}_n) &= \frac{L(\lambda \mid \mathrm{t}_1, \ldots, \mathrm{t}_n) P(\lambda \mid \nu, \mu)}{\int L(\lambda \mid \mathrm{t}_1, \ldots, \mathrm{t}_n) P(\lambda \mid \nu, \mu) \, d\lambda} \\ &= \frac{\left(\sum_{i=1}^{n} t_i^\alpha + \mu\right)^{\alpha+n}}{\Gamma(\nu + n)} \lambda^{-(\nu+n)-1} \exp\left(-\frac{1}{\lambda}\left(\sum_{i=1}^{n} t_i^\alpha + \mu\right)\right). \end{aligned} \tag{9}$$

The estimators of parameters are obtained by posterior mean, which is

$$E[P(\lambda \mid \mathrm{t}_1, \ldots, \mathrm{t}_n)] = \hat{\lambda}. \tag{10}$$

### 2.3. Markov chain Monte Carlo (MCMC)

Markov chain Monte Carlo (MCMC) is a powerful simulation technique for estimating the unknown parameters of high-dimensional probability distributions by the tools such as WinBugs and JAGS. The basic method of MCMC is the Metropolis-Hastings algorithm (MH). Gibbs sampling (GS) is a special case of the MH which generates a Markov chain by sampling from the full set of conditional distributions.

Assume that the parameters of interest and a target distribution $f(x)$ from wish we wish to generate a sample of size $N$ by the posterior distribution $f(\theta | y)$ . The algorithm is summarized as follows.

**Step 1:** Set initial values $\theta^{(0)}$

**Step 2:** Repeat the following steps

a. Set $\theta = \theta^{(t\text{-}1)}$

b. Generate new candidate parameter values $\theta'$ from a proposal distribution $q(\theta'|\theta)$ .

c. Calculate the acceptance rate

$$\alpha = \min\left(1, \frac{f(\theta'|y)q(\theta|\theta')}{f(\theta|y)q(\theta'|\theta)}\right) \tag{11}$$

d. Update $\theta^{(t)} = \theta'$ with probability $\alpha$ ; otherwise set $\theta^{(t)} = \theta$.

This study selects proposal distribution of two cases: Jeffery's prior by Eq. (12) and normal distribution by Eq. (13). Jeffery's prior distribution is given by Fisher information matrix, $I(\alpha,\lambda)$ (Zaidi, Ould Bouamama, & Tagina, 2012).

$$\pi(\alpha,\lambda) \propto \sqrt{|I(\alpha,\lambda)|}\ , \tag{12}$$

$$\pi(\alpha,\lambda) \propto p(\lambda|t,\alpha)\,p(\alpha|t,\lambda)\ , \tag{13}$$

where $\lambda \sim N(\lambda|t,\alpha)$ and $\alpha \sim N(\alpha|t,\lambda)$ .

Gibbs sampling is a special case of MH where proposal distributions result in acceptance probability $\alpha = 1$ , and therefore the sampling candidates are always accepted in all iterations. Given a particular state of the chain $\theta^{(t)}$ , we generate the new parameter values by Gibbs' sequence. It is reasonable to model of the scale parameter with a normal prior and the shape parameter with an exponential prior (Zaidi et. al., 2012).

## 3. NUMERICAL STUDY

### 3.1. Data Set

We generate $n = 10000$ lifetime observations from Weibull distribution with $\alpha = 2.5$ and $\theta = 100$ for different sample size. As shown in Figure 1, we perform the right censored simulation by changing the failure units ( $r$ ). This means that the simulation is performed by changing the proportion of censored observations from $r = 10000$ to $r = 99700$. In other words, simulation with changing the proportion of censored observations means experimenting with reducing the experiment time. Under the simulation conditions, the parameter estimation methods are compared.

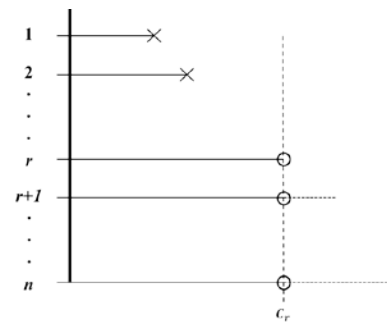


Figure 1. Right censored observations

### 3.2. Results and Discussions

To assess the accuracy of estimation methods, mean square errors (MSE) are computed, as presented in Figure 2 and Figure 3. Figure 2 and Figure 3 shows the estimators of shape and scale parameter with MLE, BA, MH and GS of MCMC.

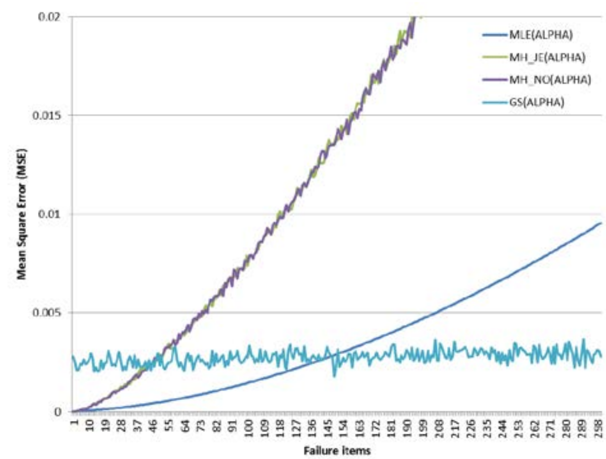


Figure 2. Mean square error of estimating shape parameter

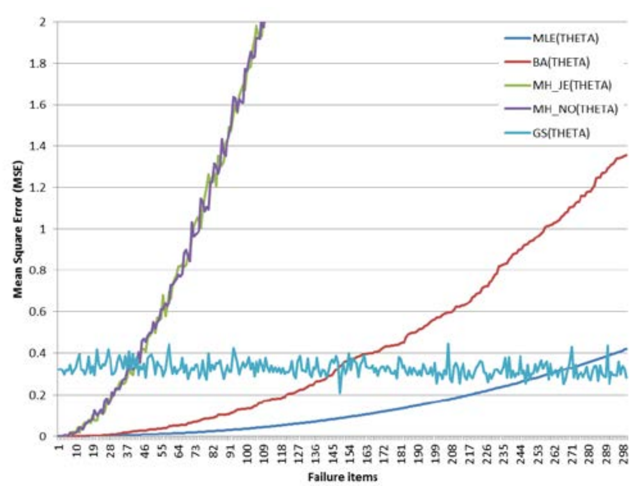


Figure 3. Mean square error of estimating shape parameter

The result has the following meanings.

- The accuracy of estimating shape parameter was better than scale parameter in every method.
- The results of both parameter estimation show "MLE > BA > MH of MCMC" except for GS of MCMC. MSE of GS is not increasing as censored observations increase. Therefore, there can be cross point between GS and other methods. As a result, MLE have a better performance when it is close to completed observation. As censored observations increase, the accuracy of GS increases.
- The more increase censored observations, the more increase MSE of estimation methods. However, there is an unusual feature that MSE of GS has a tendency of staying within a certain range regardless of increasing censored observations. Finally, this study proposes appropriate parameter estimation method considering time to experiment.

## 4. CONCLUSION

Parameter estimation is an essential procedure for safety and reliability analysis. However, there are many limitations on the experimental environment and costs in order to reliability analysis. Moreover, lifetime data are often censored. This study performed parameter estimation of Weibull distribution with censored lifetime. Parameter estimation methods of this study are maximum likelihood estimation (MLE), Bayesian estimation (BA), which is traditional parameter estimation method, and Markov chain Monte Carlo (MCMC) of which Metropolis-Hastings algorithm (MH) and Gibbs sampling (GS) are used. In order to compare parameter estimation methods, we were applied in the numerical examples following Weibull distribution.

To assess the accuracy of estimation methods, mean square errors (MSE) are computed. Accuracy of estimation methods shows "MLE > BA > MH of MCMC". Moreover, MSE of GS has a tendency of staying within a certain range regardless of increasing censored observations. MLE has a better performance when it is close to completed observation, on the other hands; GS has a better when censored observations increase. As a conclusion, this study can be helpful to set the design of experiment where trade-off between the accuracy of sampling data and costs of experiment occurs considering the characteristics of Weibull distribution with censored lifetime.

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